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A single axiom for closure operators of partially ordered sets(*)

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1. Let us recall that an operator φ of a partially ordered set P is said to be a closure operator of P, if the following conditions hold:

C1: $x \leq \varphi(x)$ *for every* $x \in P$;

C2: *if* $x \leq y$, *then* $\varphi(x) \leq \varphi(y)$;

C3: $\varphi(\varphi(x)) = \varphi(x)$ *for every* $x \in P$.

In [1], António Monteiro gave a characterization of the closure operators of the lattice $\mathscr{P}(I)$ formed by all subsets of the set I, by means of one axiom. He stated that the operator φ of $\mathscr{P}(I)$ is a closure operator, if and only if one has

$$(1) \quad Y \cup \varphi(Y) \cup \varphi(\varphi(X)) \subseteq \varphi(X \cup Y),$$

for all, X, $Y \in \mathscr{P}(I)$.

In [2], it is showed that an operator φ of the partially ordered set P is a closure operator, if and only if

$$(2) \quad x \leq \varphi(y) \text{ is equivalent to } \varphi(x) \leq \varphi(y).$$

One sees that the elements of the partially ordered set, on which the operator φ is defined, are present in both conditions (1) and (2), that is to say, neither of these conditions is intrinsic.

The purpose of this note is to formulate an intrinsic characterization of the closure operators of a partially ordered set, by using only one axiom.

2. It is well known that the set of all operators of a partially ordered set P becomes a partially ordered set, by defining

$$\varphi \leq \psi, \text{ if and only if } \varphi(x) \leq \psi(x)$$

for every $x \in P$.

The identity operator of P is denoted by ε and, if φ and ψ are operators of P, one denotes by $\varphi \circ \psi$ the operator defined by $(\varphi \circ \psi)(x) = \varphi(\psi(x))$ for every $x \in P$.

We are going to state the following

Theorem: *If* P *is a partially ordered set and* φ *is an operator of* P, *then* φ *is a closure operator, if and only*

$$(3) \quad \varepsilon \leq \psi \text{ implies } \varepsilon \leq \varphi \circ \varphi \leq \varphi \circ \psi.$$

Proof: Indeed, let φ be a closure operator of P and let us suppose that ψ is an operator of P satisfying the condition $\varepsilon \leq \psi$.

This means that

$$x \leq \psi(x) \text{ for every } x \in P,$$

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and from this it follows, by C 2,

$$\varphi(x) \leqq \varphi(\psi(x)) = (\varphi \circ \psi)(x)$$

for every $x \in P$, that is to say,

$$\varphi \leqq \varphi \circ \psi .$$

The condition C 1 and C 3 mean that one has

$$\varepsilon \leqq \varphi = \varphi \circ \varphi$$

and, consequently,

$$\varepsilon \leqq \varphi \circ \varphi \leqq \varphi \circ \psi ,$$

as wanted.

Conversely, let us suppose that condition (3) holds.

Then, since $\varepsilon \leqq \varepsilon$, one has

$$\varepsilon \leqq \varphi \circ \varphi \leqq \varphi \circ \varepsilon = \varphi .$$

This shows that one has

$$x \leqq \varphi(x) \text{ for every } x \in P ,$$

i. e., C 1 holds.

Moreover, since $\varphi \circ \varphi \leqq \varphi$, one concludes that

$$\varphi(\varphi(x)) \leqq \varphi(x) \text{ for every } x \in P .$$

On the other hand, from $t \leqq \varphi(t)$ for every $t \in P$, it follows, by putting $t = \varphi(x)$,

$$\varphi(x) \leqq \varphi(\varphi(x)) \text{ for every } x \in P ,$$

proving that C 3 holds.

Finally, let $x \leqq y$ and let ψ be the operator of P defined by the conditions

$$\begin{cases} \psi(z) = y , & \text{if } z \leqq y ; \\ \psi(z) = z , & \text{if } z \not\leqq y . \end{cases}$$

One has clearly $\varepsilon \leqq \psi$ and from this it follows, by condition (3) and C 3,

$$\varphi(x) = \varphi(\varphi(x)) \leqq \varphi(\psi(x)) = \varphi(y) .$$

Consequently, condition C 2 holds and the proof is complete.

3. It is also easy to see that the operator φ of the partially ordered set P is a closure operator, if and only if the following condition is satisfied:

(4) $\varepsilon \leqq \varphi \circ \psi$ *is equivalent to* $\varphi \leqq \varphi \circ \psi$.

In fact, if φ is a closure operator, from condition (2), it results that

$$x \leqq \varphi(\psi(x)) \text{ is equivalent to } \varphi(x) \leqq \varphi(\psi(x))$$

for every $x \in P$, and so (4) holds.

Conversely, one sees that condition (4) implies condition (2).

BIBLIOGRAPHY

[1] António Monteiro, *Caractérisation de l'opération de fermeture par un seul axiome*, Portugal. Math., **4** (1945), pp. 158-160.

[2] José Morgado, *A characterization of the closure operators by means of one axiom*, Portugal. Math., **21** (1962), pp. 155-156.