A remark on groups without center

by José Morgado
Faculdade de Ciências, Universidade do Porto — Portugal

Introduction

Let G be a multiplicative group and let e denote the neutral element of G. If the center Z of G is constituted only by the element e, then G is said to be a *group without center*.

The purpose of this note is to state the following

THEOREM: *If* G *is a group without center and the commutator subgroup* G′, *of* G, *is directly indecomposable, then* G *is directly indecomposable.*

We give two proofs of this Theorem.

For the first proof, we need some results on normal endomorphisms.

1 — Preliminary Lemmas

Let us recall that an endomorphism α of G is said to be a *normal endomorphism* of G, if one has

(1) $\alpha(x y x^{-1}) = x\,\alpha(y)\,x^{-1}$, for all $x, y \in G$.

An endomorphism α of G is called a *projector* of G, if α is normal and *idempotent* (i. e., $\alpha^2 = \alpha$).

The group G is directly indecomposable, if and only if the projectors of G are exactly the identity endomorphism ε (defined by $\varepsilon(x) = x$, for every $x \in G$) and the zero endomorphism ω (defined by $\omega(x) = e$, for every $x \in G$).

LEMMA 1: *Let* G′ *be the commutator subgroup of* G. *If* α *is a normal endomorphism of* G, *then the restriction of* α *to* G′ *is a projector of* G′.

PROOF: Indeed, let γ be the restriction of α to G'.

It is clear that γ is an endomorphism of G', for G' is a fully invariant subgroup of G.

Moreover the endomorphism γ of G' is obviously normal. Thus, in order to prove that γ is a projector of G', it is sufficient to prove that

$$\gamma^2(a) = \gamma(a), \text{ for every } a \in G'.$$

Since every $a \in G'$ is a product of commutators of G, it is clearly sufficient to prove that

$$\gamma^2(x^{-1} y^{-1} x y) = \gamma(x^{-1} y^{-1} x y), \quad \text{for all } x, y \in G.$$

By (1), one has for all $x, y \in G$,

$$\begin{aligned}\gamma^2(x^{-1} y^{-1} x y) &= \alpha^2(x^{-1} y^{-1} x y) = \\ &= \alpha(\alpha(x^{-1} y^{-1} x)\,\alpha(y)) = \\ &= \alpha(x^{-1}\,\alpha(y^{-1})\,x\,\alpha(y)) = \\ &= \alpha(x^{-1})\,\alpha(\alpha(y^{-1})\,x\,\alpha(y)) = \\ &= \alpha(x^{-1})\,\alpha(y^{-1})\,\alpha(x)\,\alpha(y) = \\ &= \alpha(x^{-1} y^{-1} x y) = \\ &= \gamma(x^{-1} y^{-1} x y),\end{aligned}$$

as wanted.

LEMMA 2: *Let* α *and* β *be normal endomorphisms of* G *and let* γ *and* δ *be, respectively, the restrictions of* α *and* β *to* G′. *If one has*

$$\gamma(a) = \delta(a), \textit{ for every } a \in G',$$

then the operator $\alpha - \beta$ *is a normal endomorphism of* G *and the subgroup* $\operatorname{Im}(\alpha-\beta)$ *is contained in the center* Z *of* G.

PROOF: In fact, from

$$\alpha(x^{-1}y^{-1}xy) = \gamma(x^{-1}y^{-1}xy) = \\ = \delta(x^{-1}y^{-1}xy) = \beta(x^{-1}y^{-1}xy),$$

for all $x, y \in G$, it follows, by the normality of α and β,

$$x^{-1}\alpha(y^{-1})\,x\,\alpha(y) = x^{-1}\beta(y^{-1})\,x\,\beta(y), \quad \text{for all } x, y \in G.$$

Hence,

$$x\,\alpha(y)\,\beta(y^{-1}) = \alpha(y)\,\beta(y^{-1})\,x, \quad \text{for all } x, y \in G, \tag{2}$$

that is to say,

$$x(\alpha-\beta)(y) = (\alpha-\beta)(y)\,x, \text{ for all } x, y \in G,$$

meaning that the set $\operatorname{Im}(\alpha-\beta)$ is contained in the center Z of G.

Now, let us see that the operator $\alpha-\beta$ is an endomorphism of G (and so $\operatorname{Im}(\alpha-\beta)$ is a subgroup of G).

One has clearly, for all $x, y \in G$,

$$(\alpha-\beta)(xy) = \alpha(xy)\,\beta(xy)^{-1} = \\ = \alpha(x)\,\alpha(y)\,\beta(y^{-1})\,\beta(x^{-1}).$$

On the other hand,

$$(\alpha-\beta)(x)\,(\alpha-\beta)(y) = \alpha(x)\,\beta(x^{-1})\,\alpha(y)\,\beta(y^{-1}).$$

Thus, one must prove that

$$\alpha(y)\,\beta(y^{-1})\,\beta(x^{-1}) = \beta(x^{-1})\,\alpha(y)\,\beta(y^{-1}), \quad \text{for all } x, y \in G$$

and this is obviously true, in view of (2).

Lastly, for all $x, y \in G$, one has, by the normality of α and β,

$$(\alpha-\beta)(xyx^{-1}) = \alpha(xyx^{-1})\,\beta(xyx^{-1})^{-1} = \\ = x\,\alpha(y)\,x^{-1}\,x\,\beta(y^{-1})\,x^{-1} = \\ = x(\alpha-\beta)(y)\,x^{-1},$$

and from here one concludes that the endomorphism $\alpha-\beta$ is normal, which completes the proof of Lemma 2.

2 — First proof of Theorem above

Let us suppose that the commutator subgroup G', of G, is directly indecomposable and let α be a normal endomorphism of G.

Then, by Lemma 1, one has necessarily either

$$\alpha(x^{-1}y^{-1}xy) = x^{-1}y^{-1}xy, \text{ for all } x, y \in G$$

or

$$\alpha(x^{-1}y^{-1}xy) = e, \text{ for all } x, y \in G.$$

This means that, if α is a normal endomorphism of G, then one has

$$\text{either } \alpha' = \varepsilon' \text{ or } \alpha' = \omega',$$

where α', ε', ω' denote, respectively, the restrictions of α, ε, ω to G'.

By Lemma 2, one has clearly

$$\text{either } \operatorname{Im}(\varepsilon-\alpha) \subseteq \{e\} \text{ or } \operatorname{Im}(\alpha-\omega) \subseteq \{e\},$$

in view of the fact that G is a group without center.

Thus, one has obviously

$$\text{either } \alpha(x) = x \text{ for every } x \in G$$
$$\text{or } \alpha(x) = e \text{ for every } x \in G,$$

meaning that

$$\text{either } \alpha = \varepsilon \text{ or } \alpha = \omega.$$

Consequently, G has only the trivial projectors, ε and ω, and so G is directly indecomposable, as it was to be shown.

3 — Another proof

Let us suppose that G is the direct product of the (normal) subgroups A and B, $G = A \times B$.

One must prove that

$$\text{either } A = e \text{ or } B = \{e\}.$$

First, we are going to show that, if $G = A \times B$, then one has $G' = A' \times B'$, G', A' and B' being, respectively, the commutator subgroups of G, A and B.

In fact, let $[g, h] = g^{-1} h^{-1} g h$ be a commutator of G.

Since $g = ab$ and $h = cd$, with $a, c \in A$ and $b, d \in B$ and, moreover, each element of A commutes with each element of B, one has

$$\begin{aligned}[g, h] &= b^{-1} a^{-1} d^{-1} c^{-1} a b c d = \\ &= a^{-1} b^{-1} d^{-1} c^{-1} a c b d = \\ &= a^{-1} c^{-1} a c b^{-1} d^{-1} b d = \\ &= [a, c][b, d].\end{aligned}$$

From this it follows that $G' \subseteq A' B'$ and, since $A' B' \subseteq G'$, one concludes that $G' = A' B'$.

Now, A' and B' are normal subgroups of G'; in fact, if $g \in G'$ and $[a, c] \in A'$, with $a, c \in A$, then

$$g[a, c] g^{-1} = [g a g^{-1}, g c g^{-1}] \in A',$$

because $g a g^{-1} \in A$ and $g c g^{-1} \in A$, proving that A' is a normal subgroup of G'. Analogously for B'.

In addition, one has $A' \cap B' \subseteq A \cap B = \{e\}$.

Consequently, $G' = A' \times B'$, as desired.

Now, since G' is directly indecomposable, one has

$$\text{either } A' = \{e\} \text{ or } B' = \{e\},$$

that is to say,

either A is an Abelian subgroup of G or B is an Abelian subgroup of G.

From this it follows that

$$\text{either } A \subseteq Z \text{ or } B \subseteq Z$$

and, since $Z = \{e\}$, one concludes that

$$\text{either } A = \{e\} \text{ or } B = \{e\},$$

as required.

BIBLIOGRAPHY

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