On symmetrical Fourier kernel I(¹)

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ABSTRACT. A generalised symmetrical FOURIER kernel has been introduced. It has been tried to give a more general form of reciprocal transform with this FOURIER kernel. Finally a formula for self-reciprocal functions associated with the H-function is being established.

1. INTRODUCTION. The functions $k(x)$ and $h(x)$ are said to form a pair of FOURIER kernels if the following pair of reciprocal equations:

$$(1.1) \qquad g(x) = \int_0^\infty k(xy) f(y)\, dy ,$$

$$(1.1)' \qquad f(x) = \int_0^\infty h(xy)\, g(y)\, dy ,$$

are simultaneously satisfied. As usual the kernels will be symmetrical if $k(x) = h(x)$ and if $k(x) \neq h(x)$ the kernels will be unsymmetrical. The functions studied by KESARWANI (1959), FOX (1961) and others as symmetrical FOURIER kernels are the G-functions.

I shall try to introduce a generalised symmetrical FOURIER kernel by taking the more general form of the H-function studied by FOX (1961). With this kernel, a new reciprocal transform has been defined. Then a formula for self-reciprocal functions associated with the H-function is given.

2. Employing the definition of the H-function, we consider the function:

$$(2.1) \qquad H^{m+n,\,p+q}_{2p+2q,\,2m+2n}(x) =$$

$$= (2\pi i)^{-1} \int_T \prod_1^m \Gamma(c_j + \gamma_j(s-1/2)) \cdot$$

$$\cdot \prod_1^p \Gamma(a_j - \alpha_j(s-1/2)) \cdot$$

$$\cdot \prod_1^n \Gamma(d_j + \delta_j(s-1/2)) \cdot$$

$$\cdot \prod_1^q (b_j - \beta_j(s-1/2)) \cdot$$

$$\cdot \left\{ \prod_1^n \Gamma(d_j - \delta_j(s-1/2)) \cdot \right.$$

$$\left. \cdot \prod_1^q \Gamma(b_j + \beta_j(s-1/2)) \right\}^{-1} \cdot$$

$$\cdot \left\{ \prod_1^m \Gamma(c_j - \gamma_j(s-1/2)) \cdot \right.$$

$$\left. \cdot \prod_1^p \Gamma(a_j + \alpha_j(s-1/2)) \right\}^{-1} \cdot x^{-s}\, ds ,$$

where the following simplifying assumptions are made:

$$(i) \qquad \gamma_j > 0 ,\, j = 1, \cdots, m ;$$

$$\alpha_j > 0 ,\, j = 1, \cdots, p ;$$

$$\delta_j > 0 ,\, j = 1, \cdots, n ;$$

$$\beta_j > 0 ,\, j = 1, \cdots, v .$$

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(*ii*)

$$D = 2\left(\sum_1^m \gamma_j - \sum_1^p \alpha_j + \sum_1^n \delta_j - \sum_1^q \beta_j\right) > 0 .$$

(*iii*) All the poles of the integrand of (2.1) are simple.

(*iv*) The contour T is a straight line parallel to the imaginary axis in the s plane and the poles of $\Gamma(c_j + \gamma_j(s-1/2))$ and $\Gamma(d_j + \delta_j(s-1/2))$ and lie to the left of T while those of $\Gamma(b_j - \beta_j(s-1/2))$ and $\Gamma(a_j - \alpha_j(s-1/2))$ lie on the right of T.

For the sake of brevity, I shall write (2.1) in the form

$$(2.2) \quad H_1(x) = (2\pi i)^{-1} \int_T M_1(s)\, x^{-s}\, ds .$$

It can be easily shown that $M_1(s)$ is the MELLIN transform of $H_1(x)$ and it satisfies the necessary and sufficient condition [9] that $H_1(x)$ may be a symmetrical FOURIER kernel is that

$$(2.3) \qquad M_1(s)\, M_1(1-s) = 1 .$$

A number of FOURIER kernels follow as particular cases by specializing the parameters in (2.1).

With the above FOURIER kernel, the new reciprocal transform may be introduced as:

$$(2.4) \qquad g(x) = \int_0^\infty H_1(xy) f(y)\, dy .$$

A systematic study of the above reciprocal transform can be made as in the case of HANKEL transforms.

The HANKEL transform introduced by VERMA [7]:

$$(2.5) \qquad g(x) = \int_0^\infty G_{2,4}^{2,1}\left(xy \,\middle|\, \begin{matrix} k-m-1/2-\nu/2,\\ -k+m+1/2+\nu/2 \\ \nu/2-\lambda-m,\ \nu/2-\lambda+m, \\ -\nu/2+\lambda+m, \\ -\nu/2+\lambda-m \end{matrix} \right) \cdot f(y)\, dy ,$$

is a special case of (2.4) for $n=0$, $v=0$, $m=2$, $p=1$, $x_j=1$, $j=1,\cdots,p$; $\gamma_j=1$, $j=1,\cdots,m$; $a_1 = k-m-1/2-\nu/2$, $a_2 = -k+m+1+\nu/2$, $c_1 = \nu/2-\lambda-m$, $c_2 = \nu/2-\lambda+m$, $c_3 = -\nu/2+\lambda+m$, $c_4 = -\nu/2+\lambda-m$ in (2.1).

The integral transform (2.5) reduces to a generalised HANKEL transform due to BHISE [2] for $\lambda = -m$, which itself reduces to HANKEL transform

$$(2.6) \qquad g(x) = \int_0^\infty (xy)^{1/2} J_\nu(xy) f(y)\, dy .$$

3. Now we estimate the asymptotic behaviour of $M_1(s)$, $s = \sigma + it$, and t real, when $|t|$ is large. For large s the asymptotic expansion of the GAMMA function is [8]:

$$(3.1) \qquad \log\Gamma(s+a) = (s+a-1/2)\log s - s + 1/2 \log(2\pi) + O(s^{-1}),$$

where $|\arg s| < \pi$. To find the behaviour of $M_1(s)$ for large $|t|$, we replace GAMMA functions involving $-s$ into those containing $+s$ with the help of the relation

$$(3.2) \qquad \Gamma(z)\,\Gamma(1-z) = \pi \operatorname{cosec} \pi z .$$

Then using (3.1) and the simplifying assumptions made in (2.1), (*i*) $\cdots$ (*iv*), we get

$$(3.3) \qquad M_1(s)\, x^{-s} = |t|^{D(\sigma-1/2)} \exp\{it(D\log|t| - \log x - B)\} \times \{Q + O(|t|^{-1})\},$$

for large $|t|$, where B is a constant and Q is also a constant but Q may have one value for large positive t and another value for large negative t.

From (3.3) it follows that if $\sigma < 1/2$, the integral (2.2) is uniformly convergent with respect to x. We may, therefore integrate through the integral sign of (2.2).

Let us take

$$(3.4) \qquad H_1^{(1)}(x) = \int_0^x H_1(x)\,dx\,,$$

then

$$(3.5) \qquad H_1^{(1)}(x) = (2\pi i)^{-1} \cdot \int_T M_1(s)(1-s)^{-1}x^{1-s}\,ds\,.$$

This has been proved to be valid only when $\sigma < 1/2$, but for $\sigma = 1/2$, the proof can be extended. On the line $\sigma = 1/2$, $M_1(s)\,x^{-s}$ is bounded from (3.3) and therefore $M_1(s)/(1-s) \in L_2(1/2 - i\infty, 1/2 + i\infty)$.

4. If $f(x) = \int_0^\infty k(xy)f(y)\,dy$, then $f(x)$ is said to be a self-reciprocal function for kernel $k(x)$. All the symmetrical FOURIER kernels can be associated with self-reciprocal functions and conversely.

Now we shall establish a formula for the self-reciprocal functions of $H_1(x)$. The following results will be required in theorem relating self-reciprocal functions. We shall write:

$$(4.1) \qquad M_1(s) = N_1(s)\,/\,P_1(s)\,,$$

where

$$(4.2) \quad N_1(s) = \prod_1^m \Gamma(c_j + \gamma_j(s-1/2)) \cdot \prod_1^p \Gamma(a_j - \alpha_j(s-1/2)) \times \prod_1^n \Gamma(d_j + \delta_j(s-1/2)) \cdot \prod_1^v \Gamma(b_j - \beta_j(s-1/2))\,.$$

Here $M_1(s)$ is the coefficient of x^{-s} in the integral (2.1) and so

$$(4.3) \qquad P_1(s) = N_1(1-s)\,.$$

THEOREM. *If*

(i) $\gamma_j > 0, j = 1, \cdots, m;\ \alpha_j > 0, j = 1, \cdots, p;\ \delta_j > 0, j = 1, \cdots, n;\ \beta_j > 0, j = 1, \cdots, v,$

(ii) $$D = 2\left(\sum_1^m \gamma_j - \sum_1^p \alpha_j + \sum_1^n \delta_j - \sum_1^v \beta_j\right) > 0\,,$$

(iii) $R(a_j) > 0, j = 1, \cdots, p;$ $R(b_j) > 0, j = 1, \cdots, v;$ $R(c_j) > 0, j = 1, \cdots, m;$ $R(d_j) > 0, j = 1, \cdots, n;$

(iv) $E_1(1/2 - s)$ *is an even function of* s,

(v) $N_1(s)\,E_1(s) \in L_2(1/2 - i\infty, 1/2 + i\infty)$,

(vi)
$$f(x) = (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} N_1(s)\,E_1(s)\,x^{-s}\,ds\,,$$

then

$$(4.4) \qquad \int_0^x f(x)\,dx = \int_0^\infty f(t)\,H_1^{(1)}(xt)\,t^{-1}\,dt\,.$$

It includes the Theorem 4 and Theorem 6 of Fox [3] as corollaries.

PROOF. This theorem is proved by performing two applications of PARSEVAL theorem, Theorem 72 [6, p. 95].

From (3.5), it follows that $M_1(s)/(1-s) \in L_2(1/2 - i\infty, 1/2 + i\infty)$ and that $H_1^{(1)}(x)/x$

is its MELLIN transform. Thus, using t as the MELLIN transform variable, it follows that $H_1^{(1)}(x)/t$, and $M_1(s)x^{1-s}/(1-s)$ are MELLIN transform of each other. Then, on using (v) and Theorem 72 [6] one can apply the PARSEVAL theorem and obtain

$$(4.5)\qquad \int_0^\infty f(t)\, H_1^{(1)}(x\,t)\, t^{-1}\, d\,t = $$

$$= (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_1(s)\, x^{1-s} (1-s)^{-1} \times$$

$$\times N_1(1-s)\, E_1(1-s)\, d\,s$$

$$(4.6)$$

$$= (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} N_1(s) E_1(s) x^{1-s} (1-s)^{-1} ds,$$

using (4.1), (4.3) and condition (iv).

Again using Theorem 72 [6] and defining the function $F(t)$, we have

$$(4.7)\qquad \int_0^x f(t)\, d\,t = \int_0^\infty f(t)\, F(t)\, d\,t$$

$$(4.8)$$

$$= (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} N_1(s) E_1(s) x^{1-s} (1-s)^{-1} ds.$$

By comparing (4.5) and (4·8), we get the required result.

The generalised H-function kernel can be utilised in the study of dual integral equations. Employing the technique [4] introduced by Fox, we can solve dual integral equations with the following H-function kernels:

$$\int_0^\infty H_{2p+2v+k,\,2m+2n+k}^{m+n,\,p+v+k}(x\,u) f(u)\, d\,u = \varphi(x),$$

$$(0 < x < 1),$$

$$\int_0^\infty H_{2p+2v+k',\,2m+2n+k'}^{m+n+k',\,p+v}(x\,u) f(u)\, d\,u = \psi(x),$$

$$(x > 1),$$

where $\varphi(x)$ and $\psi(x)$ are given and $f(x)$ is the unknown function to be found. By using fractional integration these equations can be roduced to two others with common kernel $H_{2p+2q,\,2m+2n}^{m+n,\,p+v}(x)$, which is the symmetrical FOURIER kernel (2.1).

Then $f(x)$ can be found by the known FOURIER inversion formula.

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