Some results involving G-function of two variables

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1. Introduction. In this paper we have evaluated six integrals involving G-function of two variables and LAGUERRE and HERMITE polynomials respectively. Further we have employed these integrals to establish six expansion formulae for the G-function of two variables involving LAGUERRE and HERMITE polynomials respectively. Some expansions for KAMPÉ DE FÉRIET function of two variables and MEIJER's G-function have been obtained as particular cases.

The G-function of two variables recently given by AGARWAL [1] and SHARMA [8] is a generalization of KAMPÉ DE FÉRIET's generalized hypergeometric function of two variables [2]. MEIJER's G-function, MAC-ROBERT's E-function, product of two G-functions and most of the known functions of two variables such as APPELL's functions F_1, F_2, F_3, F_4, the WHITTAKER functions of two variables and many higher transcendental functions [6, p. 215-222] may be obtained as particular cases of the G-function of two variables. Therefore, the results established in this paper are of very general character.

The modified G-function of two variables will be represented and defined as follows:

$$(1.1)\qquad G^{(m_1,m_2);\,(n_1,n_2),\,n_3}_{(p_1,p_2),\,p_3;\,(q_1,q_2),\,q_3}\left[\begin{array}{c|c} x & (a_{p_1};\ c_{p_2}) \\ & e_{p_3} \\ y & (b_{q_1};\ d_{q_2}) \\ & f_{q_3}\end{array}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j-t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}$$

$$\times\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)\,x^s y^t}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=1}^{q_3}(1-f_j+s+t)}\,ds\,dt.$$

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(b_j-s)$ $(j=1,2,\cdots,m_1)$ lie on the right and the poles of $\Gamma(1-a_j+s)$ $(j=1,2,\cdots,n_1)$ and $\Gamma(1-e_j+s+t)$ $(j=1,2,\cdots,n_3)$ to the left of the contour. Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(d_j-t)$ $(j=1,2,\cdots,m_2)$ lie to the right and the poles of $\Gamma(1-c_j+t)$ $(j=1,2,\cdots,n_2)$ and $\Gamma(1-e_j+s+t)$ $(j=1,2,\cdots,n_3)$ to the left of the contour.

Provided that $0 \leqq m_1 \leqq q_1$, $0 \leqq m_2 \leqq q_2$, $0 \leqq n_1 \leqq p_1$, $0 \leqq n_2 \leqq p_2$, $0 \leqq n_3 \leqq p_3$; the integral converges if

$$(p_3 + q_1 + q_3 + p_1) < 2(n_1 + m_1 + n_3),$$

$$(p_3 + q_2 + q_3 + p_2) < 2(m_2 + n_2 + n_3),$$

$$|\arg x| < \left[m_1 + n_1 + n_3 - \frac{1}{2}(p_3 + q_1 + q_3 + p_1)\right]\pi,$$

$$|\arg y| < \left[m_2 + n_2 + n_3 - \frac{1}{2}(p_3 + q_2 + q_3 + p_2)\right]\pi.$$

Now we discuss some important properties and particular cases of the G-function of two variables, which are apparent from the definition of the G-function of two variables.

The G-function of two variables is symmetric in parameters $a_1, \cdots, a_{n_1}$ likewise in $a_{n_1+1}, \cdots, a_{p_1}$; in $c_1, \cdots, c_{n_2}$ and $c_{n_2+1}, \cdots, c_{p_2}$; in $b_1, \cdots, b_{m_1}$ and $b_{m_1+1}, \cdots, b_{q_1}$; in $d_1, \cdots, d_{m_2}$ and $d_{m_2+1}, \cdots, d_{q_2}$ and in $e_1, \cdots, e_{m_3}$ and $e_{n_3+1}, \cdots, e_{p_3}$.

If one of the $a_j\,(j = 1, \cdots, n_1)$ is equal to one of the $b_j\,(j = m_1 + 1, \cdots, q_1)$ or one of the $b_j\,(j = 1, \cdots, m_1)$ equals one of the $a_j\,(j = n_1 + 1, \cdots, p_1)$ then each of p_1, q_1 and n_1 (and m_1) decreases by unity. This is similarly true in case of parameters $d_j s$ and $c_j s$.

Obvious changes in the variables in the integral (1. 1) give

$$(1.2) \qquad x^{\rho} y^{\sigma} G^{(m_1, m_2);\,(n_1, n_2);\,n_3}_{(p_1, p_2), p_3;\,(q_1, q_2), q_3}\left[\begin{array}{c|c} x & (a_{p_1}; c_{p_2}) \\ & e_{p_3} \\ y & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] = G^{(m_1, m_2);\,(n_1, n_2), n_3}_{(p_1, p_2), p_3;\,(q_1, q_2), q_3}\left[\begin{array}{c|c} x & (a_{p_1} + \rho; c_{p_2} + \sigma) \\ & e_{p_3} + \rho + \sigma \\ y & (b_{q_1} + \rho; d_{q_2} + \sigma) \\ & f_{q_3} + \rho + \sigma \end{array}\right].$$

$$(1.2) \qquad G^{(m_1, m_2);\,(n_1, n_2), 0}_{(p_1, p_2), p_3;\,(q_1, q_2), q_3}\left[\begin{array}{c|c} x^{-1} & (a_{p_1}; c_{p_2}) \\ & e_{p_3} \\ y^{-1} & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] = G^{(n_1, n_2), 0;\,(m_1, m_2)}_{(q_1, q_2), q_3;\,(p_1, p_2), p_3}\left[\begin{array}{c|c} x & (1 - b_{q_1}; 1 - d_{q_2}) \\ & 1 - f_{q_3} \\ y & (1 - a_{p_1}; 1 - c_{p_2}) \\ & 1 - e_{p_3} \end{array}\right].$$

Adjusting the parameters of (1. 1) as given below, we obtain the following relation between the G-function of two variables and KAMPÉ de FÉRIET function of two variables:

$$(1.3) \qquad G^{(1,1);\,(m, m), l}_{(m, m);\,(p+1, p+1), n}\left[\begin{array}{c|c} -x & (1 - b_m; 1 - c_m) \\ & 1 - a_l \\ -y & (0, 1 - e_p; 0, 1 - f_p) \\ & 1 - d_n \end{array}\right]$$

$$= \frac{\prod\limits_{j=1}^{l} \Gamma(a_j) \prod\limits_{j=1}^{m} [\Gamma(b_j)\Gamma(c_j)]}{\prod\limits_{j=1}^{n} \Gamma(d_j) \prod\limits_{j=1}^{p} [\Gamma(e_j)\Gamma(f_j)]} F\left[\begin{array}{c|c|c} l & a_1, \cdots, a_l & \\ m & b_1, c_1, \cdots, b_m, c_m & x \\ n & d_1, \cdots, d_n & y \\ p & e_1, f_1, \cdots, e_p, f_p & \end{array}\right].$$

For sake of brevity KAMPÉ de FÉRIET function may be denoted by

$$F^{l,m}_{n,p}\left[\begin{matrix}(a)_l\,;\,(b\,,c)_m\,;\,x\,,y\\ (d)_n\,;\,(e\,,f)_p\end{matrix}\right].$$

Using the definition of MEIJER'S G-function [6, p. 207, (1)], we find that the G-function of two variables can be represented by a single contour integral, viz.

$$(1.\,4)\qquad G^{(m_1,m_2);\,(n_1,n_2),n_3}_{(p_1,p_2),p_3;\,(q_1,q_2),q_3}\left[\begin{matrix}x\\ \\ y\\ \end{matrix}\middle|\begin{matrix}(a_{p_1}\,;\,c_{p_2})\\ e_{p_3}\\ (b_{q_1}\,;\,d_{q_2})\\ f_{q_3}\end{matrix}\right]=\frac{1}{2\pi i}\int_{L_1}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)}\,.$$

$$G^{m_2,\,n_2+n_3}_{p_2+p_3,\,q_2+q_3}\left[y\middle|\begin{matrix}c_1,\cdots,c_{n_2},e_1-s,\cdots,e_{p_3}-s,c_{n_2+1},\cdots,c_{p_2}\\ d_1,\cdots,d\,q_2,f_1-s,\cdots,f_{q_3}-s\end{matrix}\right]x^s\,ds\,.$$

In (1. 4), putting $m_2=q_2=1$, $d_1=0$, $n_2=n_3=p_2=p_3=0$, and $q_3=0$ and using the formula $G^{1,0}_{0,1}(z\,|\,0)=e^{-z}$, we have

$$(1.\,5)\qquad G^{(m_1,1);\,(n_1,0),0}_{(p_1,0),0;\,(q_1,1),0}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(a_{p_1}\,,-)\\ (b_{q_1}\,,0)\end{matrix}\right]=e^{-y}\cdot\frac{1}{2\pi i}\int_{L_1}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\,x^s}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)}\,ds$$

$$=e^{-y}\,G^{m_1,n_1}_{p_1,q_1}\left[x\middle|\begin{matrix}a_{p_1}\\ b_{q_1}\end{matrix}\right].$$

In (1. 5), taking $y=0$, it reduces to MEIJER'S G-function [6, p. 207, (1)] which is a generalization of many higher trascendental functions [6, pp. 215-222].

The following formulae are required in the proof

$$(1.\,6)\qquad \int_0^\infty x^{\beta-1}\,e^{-x}\,L_n^\alpha(x)\,dx=\frac{(-1)^n\,\Gamma(\beta)\,\Gamma(\beta-\alpha)}{n\,!\,\Gamma(\beta-\alpha-n)}\qquad \mathrm{Re}\,\beta>0\,,$$

which follows from [5, p. 292, (1)]

$$(1.\,7)\qquad \int_{-\infty}^\infty x^{2p}\,e^{-x^2}\,H_\nu(x)\,dx=\frac{\sqrt{\pi}\,2^{\nu-2p}\,\Gamma(2p+1)}{\Gamma(p-\nu/2+1)}\,,\qquad p=0\,,1\,,2\,,\cdots,$$

which follows from [3, p. 1, (1. 2)].

In what follows δ is a positive integer and the symbol $\Delta(\delta\,,\alpha)$ represents the set of parameters α/δ, $(\alpha+1)/\delta,\cdots$, $(\alpha+\delta-1)/\delta$.

2. The integrals. The integrals to be evaluated are

$$
(2.1)\qquad \int^{\infty} x^{\beta-1}\, e^{-x}\, L_n^{\alpha}(x)\, G^{(m_1,m_2);\,(n_1,n_2),\,n_3}_{(p_1,p_2),\,p_3;\,(q_1,q_2),\,q_3}\left[\begin{array}{c|c} y\,x^{\delta} & (a_{p_1};\,c_{p_2}) \\ & e_{p_3} \\ z\,x^{\delta} & (b_{q_1};\,d_{q_2}) \\ & f_{q_3} \end{array}\right] dx
$$

$$
= \frac{(2\pi)^{\frac{1}{2}-\frac{\delta}{2}}(-1)^n}{n!\,\delta^{\frac{1}{2}-\beta-n}}\, G^{(m_1,m_2);\,(n_1,n_2),\,n_3+2\delta}_{(p_1,p_2),\,p_3+2\delta;\,(q_1,q_2),\,q_3+\delta}\left[\begin{array}{c|c} y\,\delta^{\delta} & (a_{p_1};\,c_{p_1}) \\ & \Delta(\delta,1-\beta),\ \Delta(\delta,1+\alpha-\beta),\ e_{p_3} \\ z\,\delta^{\delta} & (b_{q_1};\,d_{q_2}) \\ & f_{q_3},\ \Delta(\delta,1+\alpha+n-\beta) \end{array}\right],
$$

$\mathrm{Re}\,(\beta+\delta\, b_j+\delta\, d_k)>0)\ \ (j=1,\cdots,m_1;\ k=1,\cdots,m_2)$,

$$
(2.2)\qquad \int_0^{\infty} x^{\beta-1}\, e^{-x}\, L_n^{\alpha}(x)\, G^{(m_1,m_2);\,(n_1,n_2),\,n_3}_{(p_1,p_2),\,p_3;\,(q_1,q_2),\,q_3}\left[\begin{array}{c|c} y\,x^{\delta} & (a_{p_1};\,c_{p_2}) \\ & e_{p_3} \\ z & (d_{q_1};\,d_{q_2}) \\ & f_{q_3} \end{array}\right] dx
$$

$$
= \frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}(-1)^n}{n!\,\delta^{1/2-\beta-n}}\, G^{(m_1,m_2);\,(2\delta+n_1,n_2),\,n_3}_{(2\delta+p_1,p_2),\,p_3;\,(\delta+q_1,q_2),\,q_3}
$$

$$
\left[\begin{array}{c|c} y\,\delta^{\delta} & [\Delta(\delta,1-\beta),\ \Delta(\delta,1+\alpha-\beta),\,a_{p_1};\,c_{p_2}] \\ & e_{p_3} \\ z & [b_{q_1},\ \Delta(\delta,1+\alpha+n-\beta);\ d_{q_2}] \\ & f_{q_3} \end{array}\right],
$$

$\mathrm{Re}\,(\beta+\delta\, b_j)>0\ \ (j=1,\cdots,m_1)$;

$$
(2.3)\qquad \int_0^{\infty} x^{\beta-1}\, e^{-x}\, L_n^{\alpha}(x)\, G^{(m_1,m_2);\,(n_1,n_2),\,n_3}_{(p_1,p_2),\,p_3;\,(q_1,q_2),\,q_3}\left[\begin{array}{c|c} y & (a_{p_1};\,c_{p_2}) \\ & e_{p_3} \\ z\,x^{\delta} & (b_{q_1};\,d_{q_2}) \\ & f_{q_3} \end{array}\right] dx
$$

$$
= \frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}(-1)^n}{n!\,\delta^{1/2-\beta-n}}\, G^{(m_1,m_2);\,(n_1,2\delta+n_2),\,n_3}_{(p_1,2\delta+p_2),\,p_3;\,(q_1,\delta+q_2),\,q_3}
$$

$$
\left[\begin{array}{c|c} y & [a_{p_1},\ \Delta(\delta,1-\beta),\ \Delta(\delta,1+\alpha-\beta),\ c_{p_2}] \\ & e_{p_3} \\ z\,\delta^{\delta} & [b_{q_1};\,d_{q_2},\ \Delta(\delta,1+\alpha+n-\beta)] \\ & f_{q_3} \end{array}\right],
$$

$\mathrm{Re}\,(\beta+\delta\, d_k)>0\ \ (k=1,\cdots,m_2)$;

(2.4)
$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_\nu(x)\, G^{(m_1, m_2);\ (n_1, n_2), n_3}_{(p_1, p_2) p_3;\ (q_1, q_2), q_3}\left[\begin{array}{c|c} y\, x^{2\delta} & (a_{p_1}; c_{p_3}) \\ & e_{p_3} \\ z\, x^{2\delta} & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] dx$$

$$= \frac{(2\pi)^{1/2-\delta/2}\, 2^\nu}{\delta^{-\nu/2-\rho}}\, G^{(m_1, m_2);\ (n_1, n_2), n_3+2\delta}_{(p_1, p_2), p_3+2\delta;\ (q_1, q_2), q_3+\delta}\left[\begin{array}{c|c} y\, \delta^{\delta} & (a_{p_1}; c_{p_2}) \\ & \Delta(2\delta, -2\rho), e_{p_3} \\ z\, \delta^{\delta} & (b_{q_1}, d_{q_2}) \\ & f_{q_3}, \Delta(\delta, \nu/2-\rho) \end{array}\right],$$

$\rho = 0, 1, 2, \cdots;$

(2.5)
$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_\nu(x)\, G^{(m_1, m_2);\ (n_1, n_2), n_3}_{(p_1, r_2), p_3;\ (q_1, q_2), q_3}\left[\begin{array}{c|c} y\, x^{2\delta} & (a_{p_1}; c_{p_2}) \\ & e_{p_3} \\ z & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] dx$$

$$= \frac{(2\pi)^{1/2-\delta/2}\, 2^\nu}{\delta^{-\nu/2-\rho}}\, G^{(m_1, m_2);\ (n_1+2\delta, n_2), n_3}_{(p_1+2\delta, p_2), p_3;\ (q_1+\delta, q_2), q_3}\left[\begin{array}{c|c} y\, \delta^{\delta} & [\Delta(2\delta, -2\rho), a_{p_1}; c_{p_2}] \\ & e_{p_3} \\ z & [b_{q_1}, \Delta(\delta, \nu/2-\rho); d_{q_2}] \\ & f_{q_3} \end{array}\right],$$

$\rho = 0, 1, 2, \cdots;$

(2.6)
$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_\nu(x)\, G^{(m_1, m_2);\ (n_1, n_2), n_3}_{(p_1, p_2), p_3;\ (q_1, q_2), q_3}\left[\begin{array}{c|c} y & (a_{p_1}; c_{p_2}) \\ & e_{p_3} \\ z\, x^{\delta} & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] dx$$

$$= \frac{(2\pi)^{1/2-\delta/2}\, 2^\nu}{\delta^{-\nu/2-\rho}}\, G^{(m_1, m_2);\ (n_1, n_2+2\delta), n_3}_{(p_1, p_2+2\delta), p_3;\ (q_1, q_2+\delta), q_3}\left[\begin{array}{c|c} y & [a_{p_1}; \Delta(2\delta, -2\rho), c_{p_2}] \\ & e_{p_3} \\ z\, \delta^{\delta} & [b_{q_1}; d_{q_2}\ \Delta(\delta, \nu/2-\rho)] \\ & f_{q_3} \end{array}\right]$$

$\rho = 0, 1, \cdots;$

where

$$(p_3 + q_1 + q_3 + p_1) < 2(n_1 + m_1 + n_3),$$

$$(p_3 + q_2 + q_3 + p_2) < 2(m_2 + n_2 + n_3),$$

$$|\arg y| < \left[m_1 + n_1 + n_3 - \frac{1}{2}(p_3 + q_1 + q_3 + p_1)\right]\pi,$$

$$|\arg z| < \left[m_2 + n_2 + n_3 - \frac{1}{2}(p_3 + q_2 + q_3 + p_2)\right]\pi.$$

PROOF. To evaluate the integral (2. 1), expressing the G-function in the integrand as (1. 1), interchanging the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j-t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}$$

$$\times\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)\,y^s z^t}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)}\int_0^\infty x^{\beta+\delta s+\delta t-1}e^{-x}L_n^\alpha(x)\,dx\cdot ds\,dt.$$

Now evaluating the inner integral with the help of (1. 6) and using multiplication formula for Gamma function [6, p. 4, (11)], we get

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j-t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}$$

$$\times\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)\,y^s z^t}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)}$$

$$\times\frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}(-1)^n\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\beta+i}{\delta}+s+t\right)\times\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\beta-\alpha+i}{\delta}+s+t\right)ds\,dt}{n!\,\delta^{\frac{1}{2}-\beta-n}\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\beta-\alpha-n+i}{\delta}+s+t\right)}.$$

On applying (1. 1) the integral is established.

Formulae (2. 2) and (2. 3) can be similarly established on applying the same procedure as above with the help of (1. 6) and formulae (2. 4), (2. 5) and (2. 6) can be similarly obtained with the help of (1. 7).

3. The Expansion Formulae.

The subject matter of expansion formulae for hypergeometric functions occupies a prominent place in the literature of special functions. The expansion formulae for hypergeometric functions of one variable were given from time to time by various mathematicians.

However, the expansions of hypergeometric functions of two variables are not much attempted so far.

The expansion formulae to be established are

(3. 1)
$$x^w\, G^{(m_1,m_2);\ (n_1,n_2),\ n_3}_{(p_1,p_2),\,p_3;\ (q_1,q_2),\,q_3}\left[\begin{array}{c|c} y\,x^\delta & (a_{p_1};\ c_{p_2}) \\ & e_{p_3} \\ & (b_{q_1};\ d_{q_2}) \\ z\,x^\delta & f_{q_3} \end{array}\right]$$

$$= \frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}}{\delta^{-1/2-\alpha-w}} \sum_{r=0}^{\infty} \frac{(-\delta)^r}{\Gamma(\alpha+r+1)}\, G^{(m_1,m_2);\ (n_1,n_2),\ n_3+2\delta}_{(p_1,p_2),\,p_3+2\delta;\ (q_1,q_2),\,q_3+\delta}$$

$$\left[\begin{array}{c|c} y\,\delta^\delta & (a_{p_1};\ c_{p_2}) \\ & \Delta(\delta,-w-\alpha),\ \Delta(\delta,-w),\ e_{p_3} \\ & (b_{q_1};\ d_{q_2}) \\ z\,\delta^\delta & f_{q_3},\ \Delta(\delta, r-w) \end{array}\right] L_r^\alpha(x),$$

$\operatorname{Re}[w+\alpha+\delta b_j+\delta d_k] > -1 \qquad (j=1,\cdots,m_1;\ k=1,\cdots,m_2);$

(3. 2)
$$x^w\, G^{(m_1,m_2);\ (n_1,n_2),\ n_3}_{(p_1,p_2),\,p_3;\ (q_1,q_2),\,q_3}\left[\begin{array}{c|c} y\,x^\delta & (a_{p_1};\ c_{p_2}) \\ & e_{p_3} \\ & (b_{q_1};\ d_{q_2}) \\ z & f_{q_3} \end{array}\right]$$

$$= \frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}}{\delta^{-1/2-\alpha-w}} \sum_{r=0}^{\infty} \frac{(-\delta)^r}{\Gamma(\alpha+r+1)}\, G^{(m_1,m_2);\ (2\delta+n_1,n_2),\ n_3}_{(2\delta+p_1,p_2),\,p_3;\ (\delta+q_1,q_2),\,q_3}$$

$$\left[\begin{array}{c|c} y\,\delta^\delta & [\Delta(\delta,-w-\alpha),\ \Delta(\delta,-w),\ a_{p_1};\ c_{p_2}] \\ & e_{p_3} \\ & [b_{q_1},\ \Delta(\delta, r-w);\ d_{q_2}] \\ z & f_{q_3} \end{array}\right] L_r^\alpha(x),$$

$\operatorname{Re}[w+\alpha+\delta b_j] > -1 \qquad (j=1,\cdots,m_1);$

(3. 3)
$$x^w\, G^{(m_1,m_2);\ (n_1,n_2),\,n_3}_{(p_1,p_2),\,p_3;\ (q_1,q_2),\,q_3}\left[\begin{array}{c|c} y & (a_{p_1};c_{p_2}) \\ & e_{p_3} \\ z\,x^\delta & (b_{q_1};d_{q_2}) \\ & f_{q_3} \end{array}\right]$$

$$= \frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}}{\delta^{-1/2-\alpha-w}} \sum_{r=0}^{\infty} \frac{(-\delta)^r}{\Gamma(\alpha+r+1)}\, G^{(m_1,m_2);\ (n_1,n_2+2\delta),\,n_3}_{(p_1,p_2+\delta),\,p_3;\ (q_1,\delta+q_2),\,q_3}$$

$$\left[\begin{array}{c|c} y & [a_{p_1};\Delta(\delta,-w-\alpha),\Delta(\delta,-w),c_{p_2}] \\ & e_{p_3} \\ z\,\delta^{\delta} & [b_{q_2};d_{q_2},\Delta(\delta,r-w)] \\ & f_{q_3} \end{array}\right] L_r^{\alpha}(x),$$

$\mathrm{Re}\,[w+\alpha+\delta\, d_j] > -1 \ (j=1,\cdots,m_2)$

$$(3.4)\qquad x^{2w}\, G^{(m_1,m_2);\,(n_1,n_2),n_3}_{(p_1,p_2),p_3;\,(q_1,q_2),q_3}\left[\begin{array}{c|c} y\,x^{2\delta} & (a_{p_1};c_{p_2}) \\ & e_{p_3} \\ z\,x^{2\delta} & (b_{q_1};d_{q_2}) \\ & f_{q_3} \end{array}\right]$$

$$\frac{\sqrt{2}\,\delta^{w}}{(2\pi)^{\delta/2}}\sum_{r=0}^{\infty}\frac{\delta^{r/2}}{r\,!}\, G^{(m_1,m_2);\,(n_1,n_2),n_3+2\delta}_{(p_1,p_2)p_3+2\delta;\,(q_1,q_2),q_3+\delta}\left[\begin{array}{c|c} y\,\delta^{\delta} & (a_{p_1};c_{p_2}) \\ & \Delta(2\delta,-2w),e_{p_3} \\ & (b_{q_1};d_{q_2}) \\ z\,\delta^{\delta} & f_{q_3},\Delta\left(\delta,\dfrac{r}{2}-w\right) \end{array}\right] H_r(\alpha),$$

$$(3.5)\qquad x^{2w}\, G^{(m_1,m_2)(n_1,n_2),n_3}_{(p_1,p_2),p_3;\,(q_1,q_2),q_3}\left[\begin{array}{c|c} y\,x^{2\delta} & (a_{p_1};c_{p_2}) \\ & e_{p_3} \\ z & (b_{q_1};d_{q_2}) \\ & f_{q_3} \end{array}\right]$$

$$=\frac{\sqrt{2}\,\delta^{w}}{(2\pi)^{\delta/2}}\sum_{r=0}^{\infty}\frac{\delta^{r/2}}{r\,!}\, G^{(m_1,m_2);\,(n_1+2\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;\,(q_1+\delta,q_2),q_3}\left[\begin{array}{c|c} y\,\delta^{\delta} & [\Delta(2\delta,-2w),a_{p_1};a_{p_2}] \\ & e_{p_3} \\ & [b_{q_1},\Delta\left(\delta,\dfrac{r}{2}-w\right);b_{q_2}] \\ z & f_{q_3} \end{array}\right] H_r(x);$$

$$(3.6)\qquad x^{2w}\, G^{(m_1,m_2);\,(n_1,n_2),n_3}_{(p_1,p_2),p_3;\,(q_1,q_2),q_3}\left[\begin{array}{c|c} y & (a_{p_1};c_{p_2}) \\ & e_{p_3} \\ z\,x^{2\delta} & (b_{q_1};d_{q_2}) \\ & f_{q_3} \end{array}\right]$$

$$=\frac{\sqrt{2}\,\delta^{w}}{(2\pi)^{\delta/2}}\sum_{r=0}^{\infty}\frac{\delta^{r/2}}{r\,!}\, G^{(m_1,m_2)(n_1,n_2+2\delta),n_3}_{(p_1,p_2+2\delta),p_3;\,(q_1,q_2+\delta),q_3}\left[\begin{array}{c|c} y & [a_{p_1};\Delta(2\delta,-2w),c_{p_2}] \\ & e_{p_3} \\ z\,\delta^{\delta} & [b_{q_1};d_{q_2},\Delta(\delta,r/2-w)] \\ & f_{q_3} \end{array}\right] H_r(x)$$

where

$$(p_3+q_1+q_3+p_1) < 2\,(n_1+m_1+n_3),$$

$$(p_3+q_2+q_3+p_2) < 2\,(m_2+n_2+n_3),$$

$$|\arg y| < \left[m_2 + n_2 + n_3 - \frac{1}{2}(p_3 + q_1 + q_3 + p_1)\right]\pi,$$

$$|\arg z| < \left[m_1 + n_1 + n_3 - \frac{1}{2}(p_3 + q_2 + q_3 + p_2)\right]\pi.$$

PROOF. To prove (3. 1), let

$$f(x) = x^w \, G^{(m_1, m_2);\ (n_1, n_2), n_3}_{(p_1, p_2), p_3;\ (q_1, q_2), q_3}\left[\begin{array}{c|c} y\,x^\delta & (a_{p_1}; c_{p_2}) \\ & e_{p_3} \\ z\,x^\delta & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] = \sum_{r=0}^{\infty} C_r\, L_r^\alpha(x). \tag{3. 7}$$

Equation (3. 7) is valid, since $f(x)$ is continuous and of bounded variation in the open interval $(0, \infty)$, when $w \geqq 0$.

Multiplying both sides of (3. 7) by $x^\alpha e^{-x} L_u^\alpha(x)$ and integrating with respect to x from 0 to ∞, we have

$$\int_0^\infty x^{w+\alpha}\, e^{-x}\, L_u^\alpha(x)\, G^{(m_1, m_2);\ (n_1, n_2), n_3}_{(p_1, p_2), p_3;\ (q_1, q_2), q_3}\left[\begin{array}{c|c} y\,x^\delta & (a_{p_1}; c_{p_2}) \\ & e_{p_3} \\ z\,x^\delta & (b_{q_1}; d_{q_2}) \\ & f_{q_3} \end{array}\right] dx$$

$$= \sum_{r=0}^{\infty} C_r \int_0^\infty x^\alpha\, e^{-x}\, L_u^\alpha(x)\, L_r^\alpha(x)\, dx.$$

Now using (2. 1) and the orthogonality property of LAGUERRE polynomials [5, p. 292-293, (2) and (3)], we obtain

$$C_u = \frac{(2\pi)^{\frac{1}{2} - \frac{1}{2}\delta}\,(-1)^u}{\Gamma(\alpha + u + 1)\,\delta^{-1/2 - \alpha - u - w}}\, G^{(m_1, m_2);\ (n_1, n_2), n_3 + 2\delta}_{(p_1, p_2), p_3 + 2\delta;\ (q_1, q_2), q_3 + \delta} \tag{3. 8}$$

$$\left[\begin{array}{c|l} y\,\delta^\delta & (a_{p_1}; c_{p_2}) \\ & \Delta(\delta, -w-\alpha)\,\Delta(\delta, -w),\, e_{p_3} \\ z\,\delta^\delta & (b_{q_1}; d_{q_2}) \\ & f_{q_3},\, \Delta(\delta, u - w) \end{array}\right].$$

From (3. 7) and (3. 8) formula (3. 1) is obtained.

The formulae (3. 2) and (3. 3) can be established on applying the same procedure as above with the help of (2. 2) and (2. 3) respectively.

To prove (3. 4), let

$$(3.\,9)\qquad f(x)=x^{2w}\,G^{(m_1,m_2);\;(n_1,n_2),n_3}_{(p_1,p_2),p_3;\;(q_1,q_2),q_3}\left[\begin{array}{c|c} y\,x^{2\delta} & (a_{p_1};c_{p_2}) \\ & e_{p_3} \\ z\,x^{2\delta} & (b_{q_1};d_{q_2}) \\ & f_{q_3}\end{array}\right]=\sum_{r=0}^{\infty} C_r\,H_r(x).$$

Equation (3. 9) is valid, since $f(x)$ is continuous and of bounded variation in the open interval $(-\infty,\infty)$, when $w \geq 0$.

Multiplying both sides of (3. 9) by $e^{-x^2}H_u(x)$, integrating with respect to x from $-\infty$ to ∞, and using the orthogonality property of HERMITE polynomials [7, pp. 192-193, (5) and (6)], we get

$$(3.\,10)\qquad C_u=\frac{\sqrt{2}\,\delta^{u/2+w}}{u\,!\,(2\,\pi)^{\delta/2}}\;G^{(m_1,m_2);\;(n_1,n_2),n_3+2\delta}_{(p_1,p_2),p_3+2\delta;\;(q_1,q_2),q_3+2\delta}\left[\begin{array}{c|c} y\,\delta^{\delta} & (a_{p_1};c_{p_2}) \\ & \Delta(2\,\delta\,,-2\,w)\,,e_{p_3} \\ z\,\delta^{\delta} & (b_{q_1};d_{q_2}) \\ & f_{q_3}\,,\;\Delta(\delta\,,u/2-w)\end{array}\right].$$

Now with the help of (3. 9) and (3. 10) the expansion (3. 4) is obtained.

The expansions (3. 5) and (3. 6) can similarly be established using the integrals (2. 5) and (2. 6) respectively.

4. Particular Cases.

On specialising the parameters, the G-function of two variables may be reduced to many functions of one and two variables. However, only a few interesting particular cases are given below.

(i) In (3. 1) and (3. 4) reducing the G-functions of two variables into KAMPÉ DE FÉRIET function of two variables in view of (3. 1), we obtain

$$(4.\,1)\qquad x^{w}\,F^{l,m}_{n,p}\left[\begin{array}{c}(a)_l\,;(b\,,c)_m\,;y\,x^{\delta}\,,z\,x^{\delta}\\ (d)_n\,;(e\,,f)_p\end{array}\right]$$

$$=\frac{(2\,\pi)^{\frac{1}{2}-\frac{1}{2}\delta}}{\delta^{-1/2-\alpha-w}}\sum_{r=0}^{\infty}\frac{\prod\limits_{i=0}^{\delta-1}\Gamma\left(\dfrac{w+1+i}{\delta}\right)\prod\limits_{i=0}^{\delta-1}\Gamma\left(\dfrac{1+\alpha+w+i}{\delta}\right)(-\delta)^r}{\Gamma(\alpha+r+1)\prod\limits_{i=0}^{\delta-1}\Gamma\left(\dfrac{1+w-r+i}{\delta}\right)}$$

$$F^{l+2\delta,m}_{n+\delta,p}\left[\begin{array}{c}(a)_l\,,\;\Delta(\delta\,,1+w+\alpha)\,,\;\Delta(\delta\,,1+w)\,;(b\,,c)_m\,;y\,\delta^{\delta}\,,\,z\,\delta^{\delta}\\ (d)_n\,,\;\Delta(\delta\,,1+w-r)\,;(e\,,f)_p\end{array}\right]L^{\alpha}_r(x)\,,$$

$\mathrm{Re}(w+\alpha)>-1\,;$

(4. 2)
$$x^{2w} F^{l,m}_{n,p}\left[\begin{matrix}(a)_l ; (b, c)_m ; y x^{2\delta}, z x^{2\delta} \\ (d)_n ; (e, f)_p\end{matrix}\right]$$

$$= \frac{\sqrt{2}\,\delta^w}{(2\pi)^{\delta/2}} \sum_{r=0}^{\infty} \frac{\delta^{r/2} \prod_{i=1}^{2\delta-1} \Gamma\left(\frac{1+2w+i}{2\delta}\right)}{r! \prod_{i=0}^{\delta-1} \Gamma\left(\frac{1+w-r/2+i}{\delta}\right)}$$

$$F^{l+2\delta,m}_{n+\delta,p}\left[\begin{matrix}(a)_l, \Delta(2\delta, 1+2w) ; (b, c)_m ; y\delta^\delta, z\delta^\delta \\ (d)_n, \Delta(\delta, 1+w-r/2) ; (e, f)_p\end{matrix}\right] H_r(x),$$

where

$$p+n < l+m+1, \quad |\arg y| \text{ and } |\arg z| < \frac{1}{2}(l+m+1-p-n)\pi .$$

Similarly other results involving Kampé de Fériet function of two variables corresponding to the formulae (2. 1) to (2. 6), (3. 2), (3. 3), (3. 5) and (3. 6) may be obtained easily.

(*ii*) In (3. 2) and (3. 5) setting the parameters in view of (1. 5), we get

(4. 1)
$$x^w G^{m_1,n_1}_{p_1,q_1}\left[y x^\delta \middle| \begin{matrix} a_{p_1} \\ b_{q_1}\end{matrix}\right] = \frac{(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}}{\delta^{-1/2-\alpha-w}} \sum_{r=0}^{\infty} \frac{(-\delta)^r}{\Gamma(\alpha+1+r)}$$

$$G^{m_1,n_1+2\delta}_{p+2\delta,q_1+\delta}\left[y\delta^\delta \middle| \begin{matrix}\Delta(\delta, -w-\alpha),\ \Delta(\delta, -w),\ a_{p_1} \\ b_{q_1},\ \Delta(\delta, r-w)\end{matrix}\right] L^\alpha_r(x),$$

where

$$p_1+q_1 < 2(m_1+n_1), \quad |\arg y| < \left(m_1+n_1-\frac{1}{2}p_1-\frac{1}{2}q_1\right)\pi,$$

$$\operatorname{Re}(w+\alpha+\delta b_j) > -1 \qquad (j = 1, \cdots, m_1).$$

Which is an expansion similar to the formula [4, p. 5, (2. 8)] recently given by the author.

(4. 2)
$$x^{2w} G^{m_1,n_1}_{p_1,q_2}\left[y x^{2\delta} \middle| \begin{matrix} a_{p_1} \\ b_{q_1}\end{matrix}\right]$$

$$= \frac{\sqrt{2}\,\delta^\omega}{(2\pi)^{\delta/2}} \sum_{r=0}^{\infty} \frac{\delta^{r/2}}{r!} G^{m_1,n_1+2\delta}_{p_1+2\delta,q_1+\delta}\left[y\delta^\delta \middle| \begin{matrix}\Delta(2\delta, -2w),\ a_{p_1} \\ b_{q_1},\ \Delta\left(\delta, \frac{r}{2}-w\right)\end{matrix}\right] H_r(x),$$

where

$$p_1 + q_1 < 2(m_1 + n_1), \quad |\arg y| < \left(m_1 + n_1 - \frac{1}{2}p_1 - \frac{1}{2}q_1\right)\pi.$$

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