

CLASSROOM NOTE**A single axiom for equivalence relations**

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1. In [1], p. 21, B. H. NEUMANN formulated the following problem: «Is there a single identity involving a binary relation ρ on S to S and $=, \circ, \cup, \cap, ^{-1}, \iota, \varepsilon, \omega$, such that ρ is an equivalence if and only if it satisfies the identity?».

The purpose of this note is to solve this problem.

2. Let us recall that a binary relation α on S to S is a subset of the cartesian product $S \times S$. Instead of $(x, y) \in \alpha$, we shall write $x \alpha y$ (read: « x stands in the relation α to y »).

The *empty relation* is denoted by ε , i. e., one has $x \varepsilon y$ for no element $(x, y) \in S \times S$. The *universal relation* is denoted by ω , i. e., one has $x \omega y$ for every element $(x, y) \in S \times S$. The *identity relation* is denoted by ι , i. e., one has $x \iota y$ if and only if $x = y$.

The *converse* α^{-1} of the binary relation α is the binary relation defined by

$$x \alpha^{-1} y \text{ if and only if } y \alpha x.$$

It is immediate that

$$(1) \quad (\alpha^{-1})^{-1} = \alpha.$$

Let α and β be binary relations on S to S . Since α and β are subsets of $S \times S$, one has

$$\alpha = \beta, \text{ if and only if } x \alpha y \text{ is equivalent to } x \beta y;$$

$\alpha \subseteq \beta$ if and only if $x \alpha y$ implies $x \beta y$;
 $x \alpha \cup \beta y$, if and only if $x \alpha y$ or $x \beta y$;
 $x \alpha \cap \beta y$, if and only if $x \alpha y$ and $x \beta y$.

It is easy to see that

$$(2) \quad \alpha \subseteq \beta \text{ implies } \alpha^{-1} \subseteq \beta^{-1}.$$

The product $\alpha \circ \beta$ of α by β is defined by the condition $x \alpha \circ \beta y$, if and only if there is some element z in S such that $x \alpha z$ and $z \beta y$.

This product is associative. Instead of $\alpha \circ \alpha$, we shall write α^2 .

One sees easily that

$$(3) \quad (\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1},$$

$$(4) \quad \alpha \subseteq \beta \text{ implies } \alpha \circ \gamma \subseteq \alpha \circ \beta \circ \gamma \text{ and } \gamma \circ \alpha \subseteq \gamma \circ \beta,$$

$$(5) \quad \gamma \circ (\alpha \cup \beta) = (\gamma \circ \alpha) \cup (\gamma \circ \beta),$$

$$(6) \quad (\alpha \cup \beta) \circ \gamma = (\alpha \circ \gamma) \cup (\beta \circ \gamma),$$

for all binary relations α, β, γ on S to S (see, for instance, [2], pp. 9-11).

3. As it is well known, the binary relation ρ is said to be

$$(i) \text{ reflexive, if } \iota \subseteq \rho,$$

$$(ii) \text{ symmetric, if } \rho^{-1} \subseteq \rho,$$

$$(iii) \text{ transitive, if } \rho^2 \subseteq \rho.$$

