CLASSROOM NOTE

A single axiom for equivalence relations

por José Morgado

Instituto de Matemática, Universidade Federal de Pernambuco, Brasil

1. In [1], p. 21, B. H. Neumann formulated the following problem: «Is there a single identity involving a binary relation ρ on S to S and $=, \circ, \cup, \cap, ^{-1}, \iota, \varepsilon, \omega$, such that ρ is an equivalence if and only if it satisfies the identity?».

The purpose of this note is to solve this problem.

2. Let us recall that a binary relation α on S to S is a subset of the cartesian product $S \times S$. Instead of $(x, y) \in \alpha$, we shall write $x \alpha y$ (read: «x stands in the relation α to y»).

The *empty relation* is denoted by ε, i. e., one has $x \varepsilon y$ for no element $(x, y) \in S \times S$. The universal relation is denoted by ω, i. e., one has $x \omega y$ for every element $(x, y) \in S \times S$. The *identity relation* is denoted by ι, i. e., one has $x \iota y$ if and only if $x = y$.

The *converse* α^{-1} of the binary relation α is the binary relation defined by

$$x \alpha^{-1} y \text{ if and only if } y \alpha x .$$

It is immediate that

$$(1) \qquad (\alpha^{-1})^{-1} = \alpha .$$

Let α and β be binary relations on S to S. Since α and β are subsets of $S \times S$, on has

$\alpha = \beta$, if and only if $x \alpha y$ is equivalent to $x \beta y$;

$\alpha \subseteq \beta$ if and only if $x \alpha y$ implies $x \beta y$;

$x \alpha \cup \beta y$, if and only if $x \alpha y$ or $x \beta y$;

$x \alpha \cap \beta y$, if and only if $x \alpha y$ and $x \beta y$.

It is easy to see that

$$(2) \qquad \alpha \subseteq \beta \text{ implies } \alpha^{-1} \subseteq \beta^{-1} .$$

The product $\alpha \circ \beta$ of α by β is defined by the condition $x \alpha \circ \beta y$, if and only if there is some element z in S such that $x \alpha z$ and $z \beta y$.

This product is associative. Instead of $\alpha \circ \alpha$, we shall write α^2.

One sees easily that

$$(3) \qquad (\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1} ,$$

$$(4) \qquad \alpha \subseteq \beta \text{ implies } \alpha \circ \gamma \subseteq \circ \beta \circ \gamma \text{ and } \gamma \circ \alpha \subseteq \gamma \circ \beta ,$$

$$(5) \qquad \gamma \circ (\alpha \cup \beta) = (\gamma \circ \alpha) \cup (\gamma \circ \beta) ,$$

$$(6) \qquad (\alpha \cup \beta) \circ \gamma = (\alpha \circ \gamma) \cup (\beta \circ \gamma) ,$$

for all binary relations α, β, γ on S to S (see, for instance, [2], pp. 9-11).

3. As it is well known, the binary relation ρ is said to be

(*i*) *reflexive*, if $\iota \subseteq \rho$,

(*ii*) *symmetric*, if $\rho^{-1} \subseteq \rho$,

(*iii*) *transitive*, if $\rho^2 \subseteq \rho$.

From (1) and (2) it follows that, if ρ is symmetric, then one has $\rho = \rho^{-1}$. If ρ is reflexive, then from $\iota \subseteq \rho$ and $\iota \circ \rho = \rho$, it follows $\rho \subseteq \rho^2$ by (4) and, consequently, if ρ is reflexive and transitive, then one has $\rho = \rho^2$.

Now, we are going to state the following

THEOREM 1: *Let ρ be a binary relation on* S *to* S. *Then ρ is an equivalence relation, if and only if one has*

$$\rho = (\iota \cup \rho^{-1})^2 . \tag{7}$$

PROOF: Indeed, let us suppose that condition (7) holds. Then, since $\iota \subseteq \iota \cup \rho^{-1}$, one has by (4)

$$\begin{aligned} \iota \cup \rho^{-1} = \iota \circ (\iota \cup \rho^{-1}) \subseteq (\iota \cup \rho^{-1}) \circ (\iota \cup \rho^{-1}) = \\ = (\iota \cup \rho^{-1})^2 = \rho \end{aligned}$$

and hence

$$\iota \subseteq \rho \text{ and } \rho^{-1} \subseteq \rho ,$$

that is to say, ρ is reflexive and symmetric.

Since $\rho^{-1} = \rho$, one has by (7)

$$(\iota \cup \rho)^2 = \rho . \tag{8}$$

From $\rho \subseteq \iota \cup \rho$, it follows by (4) and (8)

$$\rho^2 \subseteq (\iota \cup \rho) \circ \rho \subseteq (\iota \cup \rho) \circ (\iota \cup \rho) = (\iota \cup \rho)^2 = \rho ,$$

proving that ρ is also transitive and, consequently, ρ is an equivalence relation.

Conversely, let us suppose that ρ is an equivalence relation.

By (5) and (6), one has

$$\begin{aligned} (\iota \cup \rho^{-1})^2 = (\iota \cup \rho^{-1}) \circ (\iota \cup \rho^{-1}) = \\ = \iota \cup \rho^{-1} \cup \rho^{-1} \cup (\rho^{-1})^2 . \end{aligned}$$

Since $\rho^{-1} = \rho$, it results

$$(\iota \cup \rho^{-1})^2 = \iota \cup \rho \cup \rho^2$$

and since $\iota \subseteq \rho$ and $\rho^2 \subseteq \rho$, it follows

$$(\iota \cup \rho^{-1})^2 = \rho ,$$

as wanted.

Another characterization of the equivalence relations by a single identity is given by the following

THEOREM 2: *Let ρ be a binary relation on* S *to* S. *Then ρ is an equivalence relation, if and only if*

$$\rho = \iota \cup \rho^{-1} \cup \rho^2 . \tag{9}$$

PROOF: In fact, let ρ be an equivalence relation. Then, from (*i*), (*ii*) and (*iii*), one concludes that

$$\iota \cup \rho^{-1} \cup \rho^2 \subseteq \rho .$$

On the other hand, since ρ is reflexive and transitive, one has $\rho^2 = \rho$ and hence

$$\rho \subseteq \iota \cup \rho^{-1} \cup \rho^2$$

and consequently (9) holds.

Conversely, if (9) holds, then one has

$$\iota \subseteq \rho , \rho^{-1} \subseteq \rho \text{ and } \rho^2 \subseteq \rho$$

proving that ρ is an equivalence relation.

By a similar way, one states that ρ is an equivalence relation, if and only if one of the following conditions holds:

$$\rho^{-1} = (\iota \cup \rho)^2 \tag{10}$$

$$\rho^{-1} = \iota \cup \rho \cup \rho^2 \tag{11}$$

REFERENCES

[1] B. H. NEUMANN, *Special Topics in Algebra: Universal Algebra,* Courant Institut of Mathematical Sciences, New York, 1962.

[2] JOSÉ MORGADO, *Introdução à Teoria dos Reticulados,* Textos de Matemática, Recife, 1962.