Topological Semigroups

by John B. Pan, S. J.
Fu Jen University, Taipei, Formosa

Introduction.

A topological semigroup is a system consisting of a set S, an operation $\cdot$, (we omit this dot and write this operation by juxtaposition), and a topology T, satisfying the following conditions:

1) for any $x, y \in S$, $xy \in S$;

2) for $x, y, z \in S$, $(xy)z = x(yz)$;

3) the operation $\cdot$ is continuous in the topology T.

A topological subsemigroup H of a semigroup S is a topological subspace of S and also a subsemigroup of S.

An equivalence relation R defined on a semigroup S is called homomorphic if for any $a, b, c, d \in S$, aRb and cRd imply $acRbd$.

Given an homomorphic equivalence relation R on S, we call the set of equivalence classes mod R the quotient set and we denote it by S/R.

The mapping from S onto S/R defined by $n(x) =$ the class mod R to which x belongs is called the natural mapping from S onto S/R.

The family U of all subsets U^* of S/R such that $n^{-1}(U^*)$ is open in S is a topology for S/R and is called the quotient topology for S/R.

We use the term homomorphism to mean continuous homomorphism. In general, we use the terms mapping, function to mean continuous mapping, continuous function.

Let S be a semigroup, R be a homomorphic equivalence relation on S, and let S/R be the quotient set. We define an operation on S/R in the following manner. Suppose that A and B are two arbitrary elements in S/R, then $AB = C$ if for any $a \in A$ and $b \in B$ we have $ab \in C$. This operation is well-defined because R is a homomorphic equivalence relation. Also it is associative, because the semigroup S is associative. Therefore the quotient set S/R with the operation just defined is a semigroup. We call it the quotient semigroup.

We say a semigroup S satisfies the condition A if for every open set U of S, the subset $n^{-1}(n(U))$ is also open, where n is the natural mapping from S onto S/R.

In this paper we shall prove the following theorems:

Theorem 1. *If the semigroup* S *satisfies the condition* A, *then the quotient set* S/R *is a topological semigroup with the quotient topology, and the natural mapping* n *from* S *onto* S/R *is an open topological homomorphism.*

Theorem 2. *If* S *and* T *are two semigroups and* g *is a homomorphism from* S *onto* T, *then* g *induces a homomorphic equivalence relation* R_g *on* S.

Theorem 3. *Let* S *and* T *be two topological semigroups and let* g *be an open homomorphism from* S *onto* T. *Then*

a) S/R_g *is a topological semigroup with the quotient topology;*

b) *the natural mapping* n *from* S *onto* S/R_g *is an open homomorphism;*
c) *the mapping* h *from* S/R_g *onto* T *defined by* $h(A) = g(a)$ *for any* $a \in A$ *as a subset of* S *and* $A \in S/R_g$ *is a topological isomorphism.*

THEOREM 4. (The First Isomorphism Theorem). *Let* S *and* T *be two topological semigroups both satisfying the condition* A. *Let* g *be an open homomorphism from* S *onto* T *and let* R^* *be a homomorphic equivalence relation defined on* T. *Then there is a homomorphic equivalence relation* **R** *on* S *and there is a mapping* h *from* S/R *onto* T/R^* *which is a topological isomorphism.*

At the end of the paper, we give an example to illustrate the theorems.

THEOREMS

THEOREM 1. *If the semigroup* S *satisfies the condition* A, *then the quotient set* S/R *is a topological semigroup with the quotient topology, and the natural mapping* n *from* S *onto* S/R *is an open topological homomorphism.*

PROOF. We have shown that S/R is an abstract semigroup. Now we wish to show that the natural mapping n from S to S/R is an abstract homomorphism. Let X and Y be two equivalence classes mod R, and let $XY = Z$. Then by definition of the operation in S/R, for any $x \in X$ and $y \in Y$, $xy \in Z$. Since the natural mapping n assigns each element to the class it belongs, we have

$$n(X) = X,\ n(Y) = Y, \text{ and } n(xy) = n(z) = Z.$$

These equations together with the equation $XY = Z$ imply that $n(xy) = n(x)\,n(y)$. This shows that the natural mapping n is an abstract homomorphism from S onto S/R.

Now let U^* be an open set in S/R. By the definition of the quotient topology for S/R, $n^{-1}(U^*)$ is open. Hence n is continuous.

Let U be an open set in S. Since S satisfies the condition A, $n^{-1}[n(U)]$ is open. Then by the definition of the quotient topology, $n(U)$ is open.

Now we wish to show that the semigroup operation in S/R is continuous. Let A and B be two arbitrary elements in S/R such that $AB = C$. Suppose that W^* is an open neighborhood of C. Then $W = n^{-1}(W^*)$ is an open neighborhood of C, considered as a subset of S. Since the semigroup operation in S is continuous, for every $a \in A$ and every $b \in B$ such that $ab = c$, there is an open neighborhood U_a of a and an open neighborhood V_b of b such that $U_a V_b \subset W$. Choose such a neighborhood U_a for every $a \in A$ and such a neighborhood V_b for every $b \in B$. Then

$$\bigcup_{\substack{a \in A \\ b \in B}} U_a V_b = \Big[\bigcup_{a \in A} U_a\Big]\Big[\bigcup_{b \in B} V_b\Big] \subset W.$$

Now $\bigcup_{a \in A} U_a$ is an open neighborhood of A in S, and n is an open mapping. It follows that $n\Big[\bigcup_{a \in A} U_a\Big]$ is an open neighborhood of the element A in S/R. Similarly $n\Big[\bigcup_{b \in B} V_b\Big]$ is an open neighborhood of the element B in S/R. Since $\Big[\bigcup_{a \in A} U_a\Big]\Big[\bigcup_{b \in B} V_b\Big] \subset W$, we have

$$n\Big[\bigcup_{a \in A} U_a\Big]\, n\Big[\bigcup_{b \in B} V_b\Big] = n\Big[\bigcup_{a \in A} U_a \bigcup_{b \in B} V_b\Big] \subset n(W) = W^*$$

Hence we have found an open neighborhood $n\Big[\bigcup_{a \in A} U_a\Big]$ of A and an open neighborhood $n\Big[\bigcup_{b \in B} V_b\Big]$ of B such that

$$n\left[\bigcup_{a\in A} U_a\right] n\left[\bigcup_{b\in B} V_b\right] \subset W^*.$$

This shows that the semigroup operation in S/R is continuous. With this, the proof of the theorem is complete.

THEOREM 2. *If* S *and* T *are two semigroups and* g *is a homomorphism from* S *onto* T, *then* g *induces a homomorphic equivalence relation* R_g *on* S.

PROOF. We define a relation R_g on S in the following manner. Suppose that a and a^* are two elements of S, then

$a = a^* \bmod R_g$ if and only if $g(a) = g(a^*)$.

Evidently, R_g is an equivalence relation. We show that R_g is homomorphic, i. e., if $a, a^*, b, b^* \in S$ such that $a = a^* \bmod R_g$ and $b = b^* \bmod R_g$, then $ab = a^* b^* \bmod R_g$. Now $a = a^* \bmod R_g$ implies $g(a) = g(a^*)$, and $b = b^* \bmod R_g$ implies $g(b) = g(b^*)$, These two equations imply that $g(a)g(b) = g(a^*)g(b^*)$. Since g is a homomorphism, we have $g(a)g(b) = g(ab)$ and $g(a^*)g(b^*) = g(a^* b^*)$. Hence $g(ab) = g(a^* b^*)$. This means that $ab = a^* b^* \bmod R_g$. This completes the proof.

THEOREM 3. *Let* S *and* T *be two topological semigroups and let* g *be an open homomorphism from* S *onto* T. *Then*

a) S/R_g *is a topological semigroup with the quotient topology;*
b) *the natural mapping* n *from* S *onto* S/R_g *is an open homomorphism;*
c) *the mapping* h *from* S/R_g *onto* T *defined by* $h(A) = g(a)$ *for any* $a \in A$ *as a subset of* S *and* $A \in S/R_g$ *is a topological isomorphism.*

PROOF. By theorem 2, g induces a homomorphic equivalence relation R_g on S. Let S/R_g be the quotient set. Then S/R_g is a semigroup. Let n be the natural mapping from S onto S/R_g. We show that the semigroup S satisfies the condition A.

Let U be an open subset in S. Since g is an open map, $g(U)$ is open in T. Also g is continuous. Hence the subset $g^{-1}[g(U)]$ is open in S. But $g^{-1}[g(U)] = \{x \in S \mid g(x) = g(y)$ for some $y \in U\}$ and $n^{-1}[n(U)] = \{x \in S \mid g(x) = g(y)$ for some $y \in U\}$ hence $n^{-1}[n(U)] = g^{-1}[g(U)]$ and $n^{-1}[n(U)]$ is open. This shows that S satisfies the condition A.

Since S satisfies the condition A, the parts a) and b) follow from theorem 1.

Before proving part c), we wish to show that the mapping h defined in the theorem is well-defined.

Let A be any element of S/R_g and let a^* and a^{**} be any two elements of A as a subset of S. Then

$$a^* = a^{**} \bmod R_g.$$

This implies

$$g(a^*) = g(a^{**}).$$

Hence

$$h(A) = g(a^*) = g(a^{**}).$$

This shows that h is well-defined.

Also h is a one to one mapping. For each $A \in S/R_g$ there corresponds a unique value

$$h(A) = g(a)$$

in T as shown above. Now since g is a mapping from S onto T, for each $t \in T$ there is an element $a \in S$ such that $t = g(a)$, by definition of R_g, $a = b \bmod R_g$ if and only if $g(a) = g(b)$. It follows that for each $g(a) = t$, there is one and only one equivalence class $A \bmod R_g$ such that $h(A) = g(a) = t$. Hence h is a one to one mapping.

We further show that h is an algebraic homomorphism. Let A and B be any two elements in S/R_g. Then

$$h(A\,B) = g(a\,b) = g(a)\,g(b) = h(A)\,h(B),$$

where a and b are arbitrary elements of A and B respectively. This shows that h is an algebraic homomorphism.

We show also that h is continuous. Let A be an element in S/R_g such that $h(A)=t$, and let W be an open neighborhood of t. Since $h(A)=g(a)$ for every $a \in A$, and since g is continuous, for every $a \in A$, there is an open neighborhood U_a of a such that $g(U_a) \subset W$. Choose such an open neighborhood U_a for every $a \in A$. Then $\bigcup_{a \in A}(U_a)$ is a neighborhood of A in S and $n\left[\bigcup_{a \in A}(U_a)\right]$ is an open neighborhood of the element A in S/R_g. But $g\left[\bigcup_{a \in A}(U_a)\right] = h\left\{n\left[\bigcup_{a \in A}(U_a)\right]\right\} \subset W$. So for any neighborhood W of $h(A)$, we have found a neighborhood $n\left[\bigcup_{a \in A}(U_a)\right]$ of A such that $h\left\{n\left[\bigcup_{a \in A}(U_a)\right]\right\} \subset W$. This shows that h is continuous.

Finally we show that h is open. Let U^* be an open subset of S/R_g. Since the natural mapping n from S onto S/R_g is continuous, $n^{-1}(U^*)$ is an open subset in S. Also, g is an open mapping from S onto T. So $g[n^{-1}(U^*)]$ is open in T. But

$$g[n^{-1}(U^*)] = h\{n[n^{-1}(U^*)]\} = h(U^*).$$

Hence $h(U^*)$ is open in T. This shows that h is an open mapping. This completes the proof.

We can sum up theorems 1, 2 and 3 by the following form of the fundamental theorem of homomorphism of the topological semigroups:

If the semigroup S satisfies the condition A, then the quotient set S/R is a topological semigroup with the quotient topology, and the natural mapping n from S onto S/R is an open topological homomorphism. Conversely, if g is an open homomorphism from S onto a semigroup T, then T is topologically isomorphic to the quotient semigroup S/R_g, where R_g is a homomorphic equivalence relation defined by

$$a\,R_g\,b \text{ if and only if } g(a)=g(b);\quad a,b \in S$$

THEOREM 4. *(The First Isomorphism Theorem). Let* S *and* T *be two topological semigroups both satisfying the condition* A. *Let* g *be an open homomorphism from* S *onto* T *and let* R* *be a homomorphic equivalence relation defined on* T. *Then there is a homomorphic equivalence relation* R *on* S *and there is a mapping* h *from* S/R *onto* T/R* *which is a topological isomorphism.*

PROOF. Since R^* is a homomorphic equivalence relation on T, by theorem 1, T/R^* is a topological semigroup and the natural mapping n from T onto T/R^* is an open topological homomorphism. Since the mapping g from S onto T is also a homomorphism, it follows that the product mapping $n\,g$ from S onto T/R^* is also a homomorphism. We show that $n\,g$ is open. Let U be an open set in S. Since g is open, $g(U)$ is open in T. Also, n is an open map; so $n\,g(U)$ is open in T/R^*. This shows that $n\,g$ is an open topological homomorphism.

Now S and T/R^* are two topological semigroups. S satisfies the condition A, and $n\,g$ is an open topological homomorphism from S onto T/R^*. Hence, by theorem 2, $n\,g$ induces a homomorphic equivalence relation R_{ng} and T/R^*. Denote R_{ng} by R. Then we have $S/R \simeq T/R^*$. We call this isomorphism h. This completes the proof.

EXAMPLE. To illustrate some of the foregoing theorems we give the following example.

Let $(0, \infty)$ be the semigroup of positive real numbers with addition as its operation and with the usual topology as its topology. Let

$$S = [(x, y) \mid x \in (0, \infty), y \in (0 \infty)]$$

and let the vector addition be defined in S; *i. e.*,

$$(x, y) + (x^*, y^*) = (x + x^*, y + y^*).$$

The set S with the vector addition is a semigroup.

We topologize the semigroup S with the usual product topology P; *i. e.*, the family of subsets

$$B = [(U \times V) \mid U, V \text{ are open in } (0, \infty)]$$

is the base for the topology P in S.

We define a relation R on S as follows: for $(x, y), (x^*, y^*) \in S$, $(x, y) R (x^*, y^*)$ if and only if $x = x^*$. It is easy to see that this relation R is an equivalence relation, because the equation $x = x^*$ is reflexive, symmetric, and transitive, We show that the equivalence relation R is also homomorphic.

Suppose that (x_1, y_1), (x_1^*, y_1^*), (x_2, y_2), $(x_2^*, y_2^*) \in S$ such that

$$(x_1, y_1) R (x_2, y_2) \text{ and } (x_1^*, y_1^*) R (x_2^*, y_2^*).$$

Then $x_1 = x_2$ and $x_1^* = x_2^*$. From these equations we have

$$x_1 + x_1^* = x_2 + x_2^*.$$

Hence

$$(x_1 + x_1^*, y_1 + y_1^*) R (x_2 + x_2^*, y_2 + y_2^*).$$

This means that the relation is a homomorphic equivalence relation.

The equivalence classes $\bmod R$ are of the form: $\{x\} \times (0, \infty)$. We denote the set of all equivalence classes $\bmod R$ by S/R. We define an operation in S/R in the following manner. Let $\{x\} \times (0, \infty)$ and $\{y\} \times (0, \infty)$ be any two elements in S/R. Then

$$\{x\} \times (0, \infty) + \{y\} \times (0, \infty) = \{x + y\} \times (0, \infty).$$

Since for any two positive real numbers x and y the number $x + y$ is unique, the operation defined on S is well-defined. This operation is associative, because the operation of addition in the set of positive real numbers is associative. Hence the set of equivalence classes $\bmod R$ with the operation of addition is a semigroup.

We define the natural mapping n from S onto S/R by assigning each element (x, y) to the equivalence class $\{x\} \times (0, \infty)$. We show that the mapping n is an algebraic homomorphism. Let (x, y) and (x^*, y^*) be two arbitrary elements in S. Then $n(x, y) = \{x\} \times (0, \infty)$ and $n(x^*, y^*) = \{x^*\} \times (0, \infty)$ and $n[(x, y) + (x^*, y^*)] = \{x + x^*\} \times (0, \infty)$. But

$$n(x, y) + n(x^*, y^*) = \{x\} \times (0, \infty) + \{x^*\} \times (0, \infty) = \{x + x^*\} \times (0, \infty).$$

Hence

$$n(x, y) + n(x^*, y^*) = n(x, y) + (x^*, y^*).$$

This shows that the mapping n is an abstract homomorphism.

Now we topologize the semigroup S/R with the quotient topology with respect to the mapping n. That is, a subset $U \times (0, \infty)$ is open in S/R if and only if $n^{-1}[U \times (0, \infty)]$ is open in S. We observe that

$$n^{-1}[U \times (0, \infty)] = U \times (0, \infty).$$

Hence a subset $U \times (0, \infty)$ of S/R is open if and only if the U is open in the usual topology of $(0, \infty)$.

If a subset $U \times V$ is open in S, then the subset

$$n^{-1}[U \times (0, \infty)] = U \times (0, \infty)$$

is also open in S. Hence S satisfies the condition A.

We show that the natural mapping n from S onto S/R is continuous and open. Let $U \times (0, \infty)$ be an open set in S/R. Then $n^{-1}[U \times (0, \infty)]$ which equals $U \times (0, \infty)$ is open in S. Hence n is continuous. Now let $U \times V$ be an open subset of S. Then $n(U \times V) = U \times (0, \infty)$ is open in S/R according to the observation of the last paragraph. Hence n is an open mapping.

Finally we show that the semigroup operation in S/R is continuous. Let $\{x\} \times (0, \infty)$ and $\{y\} \times (0, \infty)$ be any two elements in S/R such that

$$\{x\} \times (0, \infty) + \{y\} \times (0, \infty) = \{x+y\} \times (0, \infty).$$

Let $W \times (0, \infty)$ be an open neighborhood of $\{x + y\} \times (0, \infty)$. Then since the addition is continuous in the semigroup of positive real numbers, for an open neighborhood W of $x + y$, there are open neighborhoods U of x and V of y such that $U + V \subset W$. Choose $U \times (0, \infty)$ as an open neighborhood of $\{x\} \times (0, \infty)$ and $V \times (0, \infty)$ as an open neighborhood of $\{y\} \times (0, \infty)$.

Then

$$U \times (0, \infty) + V \times (0, \infty) = (U + V) \times (0, \infty) \subset W \times (0, \infty).$$

This shows that the semigroup operation in S/R is continuous.