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On the condition $(xy)^2 = x^2 y^2$ in nonassociative rings

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1. It is well known that a group G satisfying the condition

(1) $(xy)^2 = x^2 y^2$ for all x, y in G

is necessarily commutative.

It is also well known that if G is a group such that

(2) $(xy)^i = x^i y^i$ for three consecutive integers i and for all x, y in G,

then G is necessarily commutative ([1], p. 31, exercise 4).

However, the ring-theoretic analogues of these group-theoretic results do not hold.

Thus, McCoy ([2], p. 15, example 6 and p. 31, exercise 7) gives an example of a noncommutative ring satisfying the condition (1) and E. C. Johnsen, D. L. Outcalt and A. Yaqub [3] give an example of a noncommutative ring satisfying the condition (2).

In [3], the authors prove that *if* R *is any nonassociative* (i. e., not necessarily associative) *ring with identity satisfying the condition (1), then* R *is commutative.*

The purpose of this note is to generalize this result.

2. Let us recall that an element a of a multiplicative groupoid G is said to be an *associative element*, if one has

$$ax \cdot y = a \cdot xy, \quad xa \cdot y = x \cdot ay$$

and

$$xy \cdot a = x \cdot ya$$

for all x, y in G. The element a is said to be *cancellable*, if for all x, y in G each of the equations $ax = ay$ and $xa = ya$ implies $x = y$.

We are going to state the following

Theorem: *Let* R *be any nonassociative ring satisfying the condition*

$$(\mathrm{x}\,\mathrm{y})^2 = \mathrm{x}^2\,\mathrm{y}^2 \textit{ for all } \mathrm{x}, \mathrm{y} \textit{ in } \mathrm{R}.$$

If for every $(\mathrm{x}, \mathrm{y}) \in \mathrm{R} \times \mathrm{R}$ *there is in* R *some element* e *which is associative and cancellable in the multiplicative subgroupoid of* R *generated by the set* $\{\mathrm{x}, \mathrm{y}, \mathrm{e}\}$, *then* R *is commutative.*

Proof: Indeed, one has obviously

$$(x(y+z))^2 = (xy + xz)^2 = xy \cdot xy + $$
$$+ xy \cdot xz + xz \cdot xy + xz \cdot xz$$

for all x, y, z in R.

On the other hand, by taking into account the hypothesis of the theorem, one sees that

$$(x(y+z))^2 = x^2(y+z)^2 = \\ = x^2(y^2+yz+zy+z^2) = \\ = x^2y^2 + x^2\cdot yz + x^2\cdot zy + x^2z^2.$$

Consequently, one has

$$(3) \quad x^2\cdot yz + x^2\cdot zy = xy\cdot xz + xz\cdot xy$$

for all x, y, z in R.

If one starts with $((x+z)y)^2$, one obtains, by a similar way,

$$(4) \quad xy\cdot zy + zy\cdot xy = xz\cdot y^2 + zx\cdot y^2$$

for all x, y, z in R.

By changing x to $x+z$ in (3), one gets

$$x^2\cdot yz + xz\cdot yz + zx\cdot yz + z^2\cdot yz + \\ + x^2\cdot zy + zx\cdot zy = xy\cdot xz + xy\cdot z^2 + \\ + zy\cdot xz + zy\cdot z^2 + xz\cdot xy + z^2\cdot xy$$

and, hence, it results by (3),

$$(5) \quad xz\cdot yz + zx\cdot yz + z^2\cdot yz + zx\cdot zy = \\ = xy\cdot z^2 + zy\cdot xz + zy\cdot z^2 + z^2\cdot xy$$

for all x, y, z in R.

Now, we are going to show that the element e satisfying the conditions required in the theorem, commutes with x and y.

In fact, by putting $x = z = e$ in (3), it results

$$e^2\cdot ye + e^2\cdot ey = ey\cdot e^2 + e^2\cdot ey$$

and so

$$(6) \qquad e^2\cdot ye = ey\cdot e^2.$$

Since e is associative in the multiplicative subgroupoid generated by $\{x, y, e\}$, the equality (6) implies

$$e^2y\cdot e = (ey\cdot e)e$$

and, since e is cancellable in that groupoid, it follows

$$e^2y = ey\cdot e.$$

By repeating the argument, one obtains

$$ey = ye,$$

as wanted.

Analogously, by putting $y = z = e$ in (4), one gets $ex = xe$.

If in (5) one puts $z = e$, it results by (6),

$$(7) \qquad xe\cdot ye + ex\cdot ye + ex\cdot ey = \\ = xy\cdot e^2 + ey\cdot xe + e^2\cdot xy.$$

Or, one has

$$xe\cdot ye = (xe\cdot y)e = (x\cdot ey)e = (x\cdot ye)e = \\ = (xy\cdot e)e = xy\cdot e^2,$$

by the associativity of e and commutativity of e with y.

Similarly, one sees that

$$ex\cdot ey = e^2\cdot xy$$

and, consequently, from (7) it follows

$$ex\cdot ye = ey\cdot xe$$

and, hence,

$$(ex\cdot y)e = (ey\cdot x)e.$$

Since e is cancellable, one concludes

$$ex\cdot y = ey\cdot x,$$

hence,

$$e\cdot xy = e\cdot yx,$$

that is to say,

$$xy = yx,$$

as it was to be proved.

BIBLIOGRAPHY

[1] I. N. Herstein, *Topics in Algebra*, Blaisdell, New York, 1964.

[2] Neal H. McCoy, *Introduction to Modern Algebra*, Allyn and Bacon, Boston, 1960.

[3] E. C. Johnsen, D. L. Outcalt and Adil Yaqub, *An elementary commutativity theorem for rings*, Amer. Math. Monthly, **75** (1968), pp. 288-289.