A note on the normal endomorphisms of a group

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1. It is well known that in an abelian group, for every integer n, the mapping $\bar{n} : x \longmapsto x^n$ is an endomorphism.

In [1], E. Schenkman and L. I. Wade have considered the converse question whether a group is abelian when $\bar{n}$ is an endomorphism. One knows that, if there are three consecutive integers i for which the mappings $x \longmapsto x^i$ are endomorphisms, then the group is abelian. However, from the fact that the mappings $x \longmapsto x^i$ and $x \longmapsto x^{i+1}$ are endomorphisms for some integer i, one cannot conclude that the group be abelian ([2], Exercises 4 and 5, p. 31).

Let G be a group and let $G\{n\}$ be the subgroup of G generated by all elements whose orders divide n. In [1], it is stated that

1) *if* $\bar{n}$ *is an endomorphism, then* $G/G\{n^2-n\}$ *is abelian;*

2) *if* $\bar{n}$ *is an automorphism, then* $G/G\{n-1\}$ *is abelian;*

and, consequently,

3) *if* G *has no elements whose orders divide* n^2-n *or if* G *has no elements whose orders divide* $n-1$ *when* $\bar{n}$ *is an automorphism, then* G *is abelian.*

The purpose of this note is to improve the results obtained by Schenkman and Wade.

2. Let us recall that an endomorphism α of a group G is said to be a *normal endomorphism* of G, if α commutes with every inner automorphism of G, i. e., if one has

$$\alpha(x y x^{-1}) = x \alpha(y) x^{-1}$$

for all x, y in G.

Since $(x y x^{-1})^n = x y^n x^{-1}$ for all x, y in G, one sees that, if $\bar{n}$ is an endomorphism of G, then it is necessarily a normal endomorphism.

The identity operator of G will be denoted by ε and by $\varepsilon - \alpha$ one means, as it is natural, the operator of G defined by

$$(\varepsilon - \alpha)(x) = x \alpha(x^{-1}).$$

In general, the operator $\varepsilon - \alpha$ is not an endomorphism, as one concludes from the following

Theorem 1. *Let* α *be an endomorphism of the group* G. *Then* $\varepsilon - \alpha$ *is an endomorphism, if and only if* α *is normal. Moreover, if* α *is a normal endomorphism, then the endomorphism* $\varepsilon - \alpha$ *is normal.*

Proof. Indeed, one has

$$(\varepsilon - \alpha)(x y) = x y \alpha(y^{-1} x^{-1}) = x y \alpha(y^{-1}) \alpha(x^{-1})$$

for all x, y in G.

On the other hand,

$$(\varepsilon - \alpha)(x) \cdot (\varepsilon - \alpha)(y) = x \alpha(x^{-1}) \cdot y \alpha(y^{-1}).$$

Consequently, $\varepsilon - \alpha$ is an endomorphism, if and only if

$$y \alpha(y^{-1}) \alpha(x^{-1}) = \alpha(x^{-1}) y \alpha(y^{-1}),$$

that is to say, if and only if

$$\alpha(y^{-1})\,\alpha(x^{-1})\,\alpha(y) = y^{-1}\,\alpha(x^{-1})\,y\,,$$

for all x, y in G.

This means that $\varepsilon - \alpha$ is an endomorphism, if and only if one has

$$\alpha(y^{-1}\,x^{-1}\,y) = y^{-1}\,\alpha(x^{-1})\,y \text{ for all } x, y \text{ in } G,$$

which proves the first part of the theorem.

Moreover, one has clearly, for all x, y in G,

$$\begin{aligned} y\,(\varepsilon - \alpha)\,(x)\,y^{-1} &= y\,x\,\alpha(x^{-1})\,y^{-1} = \\ &= y\,x\,y^{-1}\,y\,\alpha(x^{-1})\,y^{-1} = \\ &= y\,x\,y^{-1}\,\alpha(y\,x^{-1}\,y^{-1}) = \\ &= (\varepsilon - \alpha)\,(y\,x\,y^{-1})\,, \end{aligned}$$

proving that $\varepsilon - \alpha$ is normal.

THEOREM 2. *If α is a normal endomorphism of the group* G, *then $\alpha - \alpha^2$ is a normal endomorphism of* G *and the quotient group* $\mathrm{G}/Ker(\alpha - \alpha^2)$ *is abelian.*

PROOF. By theorem 1, the operator $\varepsilon - \alpha$ is a normal endomorphism.

It is immediate that, if α and β are normal endomorphisms, then the composite endomorphism $\alpha \circ \beta$ is also normal.

Since

$$\alpha - \alpha^2 = \alpha \circ (\varepsilon - \alpha)\,,$$

one sees that $\alpha - \alpha^2$ is a normal endomorphism.

In order to show that the quotient group $G/\mathrm{Ker}(\alpha - \alpha^2)$ is an abelian group, it is sufficient to show that all commutators of G are in the kernel of the endomorphism $\alpha - \alpha^2$, that is to say, for all x, y in G,

$$(\alpha - \alpha^2)\,(x\,y\,x^{-1}\,y^{-1}) = e\,,$$

where e denotes the neutral element of G.

Or, by the normality of α, one has obviously

$$\begin{aligned} \alpha(x\,y\,x^{-1}\,y^{-1}) &= \alpha(x)\,\alpha(y)\,\alpha(x^{-1})\,\alpha(y^{-1}) = \\ &= \alpha(x)\,\alpha(\alpha(y)\,x^{-1}\,\alpha(y^{-1})) = \\ &= \alpha(x)\,\alpha^2(y)\,\alpha(x^{-1})\,\alpha^2(y^{-1}) = \\ &= \alpha(\alpha(x)\,\alpha(y)\,\alpha(x^{-1}))\,\alpha^2(y^{-1}) = \\ &= \alpha^2(x)\,\alpha^2(y)\,\alpha^2(x^{-1})\,\alpha^2(y^{-1}) = \\ &= \alpha^2(x\,y\,x^{-1}\,y^{-1})\,. \end{aligned}$$

From this it follows

$$\alpha(x\,y\,x^{-1}\,y^{-1})\,\alpha^2((x\,y\,x^{-1}\,y^{-1})^{-1}) = e$$

and, consequently,

$$(\alpha - \alpha^2)\,(x\,y\,x^{-1}\,y^{-1}) = e\,,$$

as it was to be proved.

COROLLARY 1. *If α is an injective normal endomorphism of the group* G, *then the quotient group* $\mathrm{G}/\mathrm{Ker}(\varepsilon - \alpha)$ *is abelian.*

In fact, from

$$\alpha(x\,y\,x^{-1}\,y^{-1}) = \alpha^2(x\,y\,x^{-1}\,y^{-1})\,,$$

it results, since α is injective,

$$x\,y\,x^{-1}\,y^{-1} = \alpha(x\,y\,x^{-1}\,y^{-1})$$

and hence

$$(\varepsilon - \alpha)\,(x\,y\,x^{-1}\,y^{-1}) = e$$

for all x, y in G, proving that $G/Ker(\varepsilon - \alpha)$ is abelian.

COROLLARY 2. *If the mapping* $\bar{\mathrm{n}} : \mathrm{x} \longmapsto \mathrm{x}^{\mathrm{n}}$ *is an endomorphism of the group* G, *then the quotient group* $\mathrm{G}/\mathrm{G}\,\{\mathrm{n}^2 - \mathrm{n}\}$ *is abelian. If, moreover, the endomorphism* $\bar{\mathrm{n}}$ *is injective, then* $\mathrm{G}/\mathrm{G}\,\{\mathrm{n} - 1\}$ *is abelian.*

In fact, $\bar{n}$ is normal and one has clearly

$$G\,\{n^2 - n\} = Ker\,(\bar{n} - \bar{n}^2)$$

and, if $\bar{n}$ is injective, then

$$G\,\{n - 1\} = Ker\,(1 - \bar{n})\,,$$

where $\bar{1}$ denotes the identity endomorphism.

3. Now, let us suppose that the endomorphism α is such that

$$x\,\alpha(x^{-1}) \in Z \text{ for every } x \in G,$$

where Z denotes the center of G.

It is easy to see that $\varepsilon - \alpha$ is normal.

Indeed, since $\alpha(x^{-1}) = x^{-1} z$ for some $z \in Z$, one has

$$\alpha(x\,y\,x^{-1}) = \alpha(x)\,\alpha(y)\,\alpha(x^{-1}) = \\ = z^{-1}\,x\,\alpha(y)\,x^{-1}\,z = x\,\alpha(y)\,x^{-1}$$

for all x, y in G, proving that α is normal.

Then, by Theorem 1, one concludes that $\varepsilon - \alpha$ is a normal endomorphism of G.

We are going to see that the quotient group $G/Ker(\varepsilon - \alpha)$ is abelian.

In fact, since

$$(\varepsilon-\alpha)(x\,y\,x^{-1}\,y^{-1}) = x\,y\,x^{-1}\,y^{-1}\,\alpha(y\,x\,y^{-1}\,x^{-1}) = \\ = x\,y\,x^{-1}\,y^{-1}\,\alpha(y)\,\alpha(x)\,\alpha(y^{-1})\,\alpha(x^{-1}) = \\ = x\,y\,x^{-1}\,\alpha(x)\,y^{-1}\,\alpha(y)\,\alpha(y^{-1})\,\alpha(x^{-1}) = \\ = x\,y\,x^{-1}\,\alpha(x)\,y^{-1}\,\alpha(x^{-1}) = \\ = x\,y\,y^{-1}\,x^{-1}\,\alpha(x)\,\alpha(x^{-1}) = \\ = e$$

for all x, y in G, one sees that the kernel of $\varepsilon - \alpha$ contains the subgroup generated by the commutators and, therefore, the quotient group $G/Ker(\varepsilon - \alpha)$ is abelian.

Conversely, let us suppose that $\varepsilon - \alpha$ is a normal endomorphism and $G/Ker(\varepsilon - \alpha)$ is abelian.

Then, one has

$$x\,y\,x^{-1}\,y^{-1}\,\alpha(y)\,\alpha(x)\,\alpha(y^{-1})\,\alpha(x^{-1}) = e$$

for all x, y in G.

From this it follows

$$x\,y\,x^{-1}\,y^{-1}\,\alpha(y) = \alpha(x)\,\alpha(y)\,\alpha(x^{-1}) = \\ = \alpha(x\,y\,x^{-1}) = \\ = x\,\alpha(y)\,x^{-1},$$

in view of the fact that α is a normal endomorphism, by Theorem 1 and $\alpha = \varepsilon - (\varepsilon - \alpha)$.

Hence

$$y\,x^{-1}\,y^{-1}\,\alpha(y) = \alpha(y)\,x^{-1}.$$

Consequently,

$$x^{-1}\,y^{-1}\,\alpha(y) = y^{-1}\,\alpha(y)\,x^{-1}$$

for all x, y in G.

This means that $y^{-1}\alpha(y) \in Z$ for every $y \in G$.

In short, the following holds:

THEOREM **3.** *If α is an endomorphism of the group G such that $x\,\alpha(x^{-1})$ is in the center of G for every $x \in G$, then $\varepsilon - \alpha$ is a normal endomorphism and $G/\mathrm{Ker}(\varepsilon - \alpha)$ is abelian; conversely, if $\varepsilon - \alpha$ is a normal endomorphism and $G/\mathrm{Ker}(\varepsilon - \alpha)$ is abelian, then α is an endomorphism such that $x\,\alpha(x^{-1})$ is in the center of G for every $x \in G$.*

In particular, one has the following

COROLLARY. *If α is a central endomorphism of the group G, i. e., if $\alpha(x) \in Z$ for every $x \in G$, then the quotient group $G/\mathrm{Ker}(\alpha)$ is abelian.*

Indeed, it is immediate that α is a normal endomorphism and so $\varepsilon - \alpha$ is also a normal endomorphism.

One has, for every $x \in G$,

$$\alpha(x) = x\,x^{-1}\,\alpha(x) = x \cdot (\varepsilon - \alpha)(x^{-1}) \in Z$$

and the conclusion follows immediately from Theorem 3.

BIBLIOGRAPHY

[1] EUGENE SCHENKMAN and L. I. WADE, *The mapping which takes each element of a group onto its nth power*, Amer. Math. Monthly, **65** (1958), pp. 33-34.

[2] I. N. HERSTEIN, *Topics in Algebra*, Blaisdell Publishing Company, 1964.

[3] H. J. ZASSENHAUS, *The Theory of Groups*, second edition, Chelsea Publishing Company, New York, 1958.