

A theorem on abelian quotient groups of a group

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1. In a previous paper [1], we have shown that, if α is a normal endomorphism of the group G , then the operator $\alpha - \alpha^2$ is also a normal endomorphism of G and the quotient group $G/\text{Ker}(\alpha - \alpha^2)$ is an abelian group.

In this note, we extend this result; we state a necessary and sufficient condition in order that the quotient group $G/\text{Ker}(\beta - \beta\alpha)$ be an abelian group, α and β being endomorphisms of G and $\beta\alpha$ being the composite of β and α .

2. It is well known that, if α and β are endomorphisms of the group G , then the operator $\beta - \alpha$ of G , defined by

$$(\beta - \alpha)(x) = \beta(x)\alpha(x^{-1}) \text{ for every } x \in G$$

need not be an endomorphism of G .

In fact, since

$$\begin{aligned} (\beta - \alpha)(xy) &= \beta(xy)\alpha(y^{-1}x^{-1}) = \\ &= \beta(x)\beta(y)\alpha(y^{-1})\alpha(x^{-1}) \end{aligned}$$

and, on the other hand,

$$(\beta - \alpha)(x)(\beta - \alpha)(y) = \beta(x)\alpha(x^{-1})\beta(y)\alpha(y^{-1}),$$

one concludes that $\beta - \alpha$ is an endomorphism, if and only if one has

$$\beta(y)\alpha(y^{-1})\alpha(x^{-1}) = \alpha(x^{-1})\beta(y)\alpha(y^{-1})$$

for all x, y in G .

This means that the following holds:

LEMMA. *If α and β are endomorphisms of the group G , then the operator $\beta - \alpha$ is an endomorphism of G , if and only if the*

image of $\beta - \alpha$ is in the centralizer of the image of α in G .

THEOREM. *Let α and β be endomorphisms of the group G . Then the operator $\beta - \beta\alpha$ is an endomorphism of G and the quotient group $G/\text{Ker}(\beta - \beta\alpha)$ is abelian, if and only if the image of $\beta - \beta\alpha$ is contained in the center of the image of β .*

PROOF. Let $\beta - \beta\alpha$ be an endomorphism. Then, as it is well known, the quotient group $G/\text{Ker}(\beta - \beta\alpha)$ is abelian, if and only if the kernel of the endomorphism $\beta - \beta\alpha$ contains the commutator subgroup of G , that is to say,

$$(1) \quad (\beta - \beta\alpha)(xyx^{-1}y^{-1}) = e$$

for all x, y in G , e being the neutral element of G .

First, let us suppose that one has

$$(2) \quad \text{Im}(\beta - \beta\alpha) \subseteq \text{Center of Im}(\beta).$$

Since

$\text{Center of Im}(\beta) \subseteq \text{Centralizer of Im}(\beta\alpha)$ in G ,

one concludes by Lemma above that the operator $\beta - \beta\alpha$ is an endomorphism of G .

Furthermore, one has

$$\begin{aligned} (\beta - \beta\alpha)(xyx^{-1}y^{-1}) &= \\ &= \beta(xyx^{-1}y^{-1})\beta\alpha(xyx^{-1}y^{-1})^{-1} = \\ &= \beta(x)\beta(y)\beta(x^{-1})\beta(y^{-1}) \cdot \\ &\cdot \beta\alpha(y)\beta\alpha(x)\beta\alpha(y^{-1})\beta\alpha(x^{-1}) = \\ &= \beta(x)\beta(y)\beta(y^{-1})\beta\alpha(y)\beta(x^{-1}) \cdot \\ &\cdot \beta\alpha(x)\beta\alpha(y^{-1})\beta\alpha(x^{-1}) = \\ &= \beta(x)\beta(x^{-1})\beta\alpha(x)\beta\alpha(y) \cdot \\ &\cdot \beta\alpha(y^{-1})\beta\alpha(x^{-1}) = e. \end{aligned}$$

proving (1).

