On Products of Generalized Hypergeometric Functions(*)

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1. Introduction. In this paper we have established a formula for product of MEIJER's G-functions. From this formula we have deduced formulae for product of MACROBERT's E-functions and generalized hypergeometric functions. In section 3, we have obtained some hypergeometric transformations and formulae on the sum of ${}_4F_3(1)$. The results established are of general character and a number of known results follow as their particular cases.

2. Product of G-functions. (1)

If $t \leqq u$, $v \leqq w$, $|\lambda x| < 1$, $|\mu y| < 1$, then

$$(2.1)\quad G_{t,u}^{f,g}\left(\lambda x \,\middle|\, \begin{matrix} {}_1\alpha_t \\ {}_1\beta_u \end{matrix}\right) G_{v,w}^{k,l}\left(\mu y \,\middle|\, \begin{matrix} {}_1\gamma_v \\ {}_1\delta_w \end{matrix}\right) = \sum_{h=1}^{f} \sum_{h'=1}^{k} A(h) B(h') \sum_{m=0}^{\infty} \frac{\prod\limits_{j=1}^{t} (1+\beta_h-\alpha_j)_m}{\prod\limits_{j=1}^{u}{}'(1+\beta_h-\beta_j)_m m!} \{\lambda x (-1)^{t-f-g}\}^m$$

$$\times {}_{u+v}F_{w+t-1}\left[\begin{matrix} 1+\delta_{h'}-{}_1\gamma_v,\ {}_1\beta_u \overset{\cdot\times\cdot}{-} \beta_h - m,\ -m;\ \dfrac{\mu y}{\lambda x}(-1)^{v+g+f-k-l-u} \\ 1+\delta_{h'} \overset{\cdot\times\cdot}{-} {}_1\delta_w,\ {}_1\alpha_t - \beta_h - m \end{matrix}\right].$$

$$(2.2)\quad G_{t,u}^{f,g}\left(\lambda x \,\middle|\, \begin{matrix} {}_1\alpha_t \\ {}_1\beta_u \end{matrix}\right) G_{v,w}^{k,l}\left(\mu y \,\middle|\, \begin{matrix} {}_1\gamma_v \\ {}_1\delta_w \end{matrix}\right) = \sum_{h=1}^{f} \sum_{h'=1}^{k} A(h) B(h') \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{v} (1+\delta_{h'}-\gamma_i)_n}{\prod\limits_{i=1}^{w}{}' (1+\delta_{h'}-\delta_i)_n\, n!}$$

$$\cdot \{\mu y (-1)^{v-k-l}\}^n \times {}_{t+w}F_{u+v-1}\left[\begin{matrix} 1+\beta_h-{}_1\alpha_t,\ {}_1\delta_w \overset{\cdot\times\cdot}{-} \delta_{h'} - n,\ -n;\ \dfrac{\lambda x}{\mu y}(-1)^{t+k+l-f-g-w} \\ 1+\beta_h \overset{\cdot\times\cdot}{-} {}_1\beta_u,\ {}_1\gamma_v - \delta_{h'} - n \end{matrix}\right].$$

(*) Por razões de natureza técnica derivadas das expressões utilizadas nestes dois artigos, resolveu a Redacção modificar a disposição gráfica dos mesmos.

(1) For the sake of brevity the symbol ${}_1a_p$ is used to denote $a_1, \cdots, a_p$; $1+\delta_h-{}_1\gamma_v$ is used to denote $1+\delta_h-\gamma_1, \cdots, 1+\delta_h-\gamma_v$ and ${}_1\beta_u \overset{\cdot\times\cdot}{-} \beta_h - m$ is used to denote $\beta_1-\beta_h-m, \cdots \cdot\times\cdot \cdots, \beta_u-\beta_h-m$.

Where

$$A(h)=\frac{\prod\limits_{j=1}^{f}{}' \Gamma(\beta_j-\beta_h)\prod\limits_{j=1}^{g}\Gamma(1+\beta_h-\alpha_j)(\lambda x)^{\beta_h}}{\prod\limits_{j=f+1}^{u}\Gamma(1+\beta_h-\beta_j)\prod\limits_{j=g+1}^{t}\Gamma(\alpha_j-\beta_h)},$$

$$B(h')=\frac{\prod\limits_{i=1}^{k}{}' \Gamma(\delta_i-\delta_{h'})\prod\limits_{i=1}^{l}\Gamma(1+\delta_{h'}-\gamma_i)(\mu y)^{\delta_{h'}}}{\prod\limits_{i=k+1}^{w}\Gamma(1+\delta_{h'}-\delta_i)\prod\limits_{i=l+1}^{v}\Gamma(\gamma_i-\delta_{h'})}.$$

PROOF. To prove this, substituting in the left hand side from [4, p. 208, (5)] (²), and expressing the hypergeometric functions as a product of two series, we get

$$(2.\,3)\qquad \sum_{h=1}^{f}\sum_{h'=1}^{k}A(h)B(h')\sum_{m=0}^{\infty}\frac{\prod\limits_{j=1}^{t}(1+\beta_h-\alpha_j)_m}{\prod\limits_{j=1}^{u}{}'(1+\beta_h-\beta_j)_m\, m!}\{\lambda x(-1)^{t-f-g}\}^m$$

$$\times\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{v}(1+\delta_{h'}-\gamma_j)_n}{\prod\limits_{j=1}^{w}{}'(1+\delta_{h'}-\delta_j)_n\, n!}\{\mu y(-1)^{v-k-l}\}^n.$$

Now with the help of [8, p. 56, (1)] and [8, p. 32, (8)], the expression (2. 3) becomes

$$\sum_{h=1}^{f}\sum_{h'=1}^{k}A(h)B(h')\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{t}(1+\beta_h-\alpha_j)_m}{\prod\limits_{j=1}^{u}{}'(1+\beta_h-\beta_j)_m\, m!}\{\lambda x(-1)^{t-f-g}\}^m$$

$$\times\frac{\prod\limits_{j=1}^{v}(1+\delta_{h'}-\gamma_j)_n\prod\limits_{j=1}^{u}{}'(\beta_j-\beta_h-m)_n(-m)_n}{\prod\limits_{j=1}^{w}{}'(1+\delta_{h'}-\delta_j)_n\prod\limits_{j=1}^{t}(\alpha_j-\beta_h-m)_n\, n!}\left\{\frac{\mu y}{\lambda x}(-1)^{v+g+f-k-l-u}\right\}^n,$$

which yields the result (2. 1).

Similarly the result (2. 2) can be obtained from (2. 3), by first summing over m and then over n, with the help of [8, p. 56, (1)].

(²) Replacing the G-functions, with the help of [4, p. 208, (6)], we obtain the same results (2. 1) and (2. 2) by virtue of [4, p. 209, (9)], when the parameters are adjusted suitably.

3. Special Cases.

(i) *Product of* E*-functions.*

In (2. 1) and (2. 2), putting $f=k=1$, $t=g=p$, $v=l=r$, $u=q+1$, $w=s+1$, using [4, p. 209, (9)] and [5, p. 439, (5)], and setting $1+\beta_1-{}_1\alpha_p={}_1a_p$, $1+\beta_1-{}_2\beta_{q+1}={}_1b_q$, $1-\delta_1-{}_1\gamma_r={}_1c_r$, $1+\delta_1-{}_2\delta_{s+1}={}_1d_s$, replacing $\frac{1}{\lambda x}$ by λx and $\frac{1}{\mu y}$ by μy, we have

$$(3.\ 1)\qquad E\begin{pmatrix}{}_1a_p\,;\lambda x\\ {}_1b_q\end{pmatrix}E\begin{pmatrix}{}_1c_r\,;\mu y\\ {}_1d_r\end{pmatrix}=\frac{\prod_{j=1}^{r}\Gamma(c_j)}{\prod_{j=1}^{s}\Gamma(d_j)}\sum_{m=0}^{p}\frac{\prod_{j=1}^{p}\Gamma(a_j+m)(-\lambda x)^{-m}}{\prod_{j=1}^{q}\Gamma(b_j+m)\,m!}$$

$$\cdot\ {}_{r+q+1}F_{s+p}\left[\begin{matrix}{}_1c_r, 1-{}_1b_q-m, -m\\ {}_1d_s, 1-{}_1a_p-m,\end{matrix}\ ;\frac{\lambda x}{\mu y}(-1)^{p-q-1}\right];$$

$$(3.\ 2)\qquad E\begin{pmatrix}{}_1a_p\,;\lambda x\\ {}_1b_q\end{pmatrix}E\begin{pmatrix}{}_1c_r\,;\mu y\\ {}_1d_s\end{pmatrix}=\frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}\Gamma(c_j+n)(-\mu y)^{-n}}{\prod_{j=1}^{s}\Gamma(d_j+n)\,n!}$$

$$\cdot\ {}_{p+s+1}F_{q+r}\left[\begin{matrix}{}_1a_p, 1-{}_1d_s-n, -n\\ {}_1b_q, 1-{}_1c_r-n\end{matrix}\ ;\frac{\mu y}{\lambda x}(-1)^{r-s-1}\right],$$

where

$$p\leq q+1,\ r\leq s+1,\ |\lambda x|>1,\ |\mu y|>1.$$

(ii) *Product of generalized hypergeometric functions.*

In (2. 1) and (2. 2), on taking $f=k=1$, $t=g=p$, $v=l=r$, $u=q+1$, $w=s+1$, using [5, p. 439, (3)], and setting $1+\beta_1-{}_1\alpha_p={}_1a_p$, $1+\beta_1-{}_2\beta_{q+1}={}_1b_q$, $1+\delta_1-{}_1\gamma_r={}_1c_r$, $1+\delta_1-{}_2\delta_{s+1}={}_1d_s$, $\lambda=-A$, $\mu=-B$, we get

$$(3.\ 3)\qquad {}_pF_q\begin{pmatrix}{}_1a_p\,;Ax\\ {}_1b_q\end{pmatrix}{}_rF_s\left[\begin{matrix}{}_1c_r\,;By\\ {}_1d_s\end{matrix}\right]=\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_m(Ax)^m}{\prod_{j=1}^{q}(b_j)_m\,m!}$$

$$\cdot\ {}_{r+q+1}F_{s+p}\left[\begin{matrix}{}_1c_r, 1-{}_1b_q-m, -m\\ {}_1d_s, 1-{}_1a_p-m\end{matrix}\ ;\frac{By}{Ax}(-1)^{p-q-1}\right];$$

$$(3.4)\qquad {}_pF_q\left({}_1a_p;\,Ax \atop {}_1b_q\right){}_rF_s\left({}_1c_r;\,By \atop {}_1d_s\right)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(c_j)_n(By)^n}{\prod_{j=1}^{s}(d_j)_n\,n!}$$

$${}_{p+s+1}F_{q+r}\left[{{}_1a_p\,,1-{}_1d_s-n\,,-n\,;\ \dfrac{Ax}{By}(-1)^{r-s-1}} \atop {{}_1b_q\,,1-{}_1c_r-n}\right],$$

where

$$p\leqq q+1\,,\ r\leqq s+1\,,\ |Ax|<1\,,\ |By|<1\,.$$

With $y=x$ (3.3) is a known result [6, p. 395, (3.5)] obtained by FIELDS and WIMP, using the techniques of Laplace transform.

4. Some hypergeometric transformations and formulae on the sum of well-poised and nearly-poised ${}_4F_3(1)$.

(*i*) In (3.3) and (3.4), putting $y=x$ and comparing the coefficients of x^n, we obtain the interesting transformation

$$(4.1)\qquad \frac{\prod_{j=1}^{p}(a_j)_n A^n}{\prod_{j=1}^{q}(b_j)_n}\,{}_{r+q+1}F_{s+p}\left[{{}_1c_r\,,1-{}_1b_q-n\,,-n\,;\ \dfrac{B}{A}(-1)^{p-q-1}} \atop {{}_1d_s\,,1-{}_1a_p-n}\right]$$

$$=\frac{\prod_{j=1}^{r}(c_j)_n B^n}{\prod_{j=1}^{s}(d_j)_n}\,{}_{p+s+1}F_{q+r}\left[{{}_1a_p\,,1-{}_1d_s-n\,,-n\,;\ \dfrac{A}{B}(-1)^{r-s-1}} \atop {{}_1b_q\,,1-{}_1c_r-n}\right].$$

Putting $A=-Z$, $B=1$, $s=r=0$ in (4.1), it reduces to a known result [6, p. 395, (3.8)].

(*ii*) In (4.1), putting $A=B=1$, $a_1=\gamma-\alpha-\beta$, $c_1=2\alpha$, $c_2=2\beta$, $d_1=2\gamma$, and using [1, 10.1, (1)] for the left side hypergeometric function, we get

$$(4.2)\qquad {}_3F_2\left[{\gamma-\alpha-\beta\,,1-2\gamma-n\,,-n\,;} \atop {1-2\alpha-n\,,1-2\beta-n}\right]=\frac{(2\gamma)_n(\gamma-\alpha)_n(\gamma-\beta)_n}{(2\alpha)_n(2\beta)_n(\gamma)_n}$$

$$\cdot\,{}_4F_3\left[{\alpha\,,\beta\,,\dfrac{1}{2}-\gamma-n\,,-n\,;} \atop {\gamma+1/2\,,1+\alpha-\gamma-n\,,1+\beta-\gamma-n}\right].$$

With $\gamma = \alpha + \beta$, it reduces to a known result obtained by Chaundy [3, (10)].

(*iii*) In (4. 1), on taking $A = B = 1$, $a_1 = \frac{1}{2} - \alpha - \beta + \gamma$, $c_1 = 2\alpha$, $c_2 = 2\beta$, $d_1 = 2\gamma$, and using [1, 10. 1, (2)], we have

$$(4.\,3) \quad {}_3F_2\left[\begin{matrix}1/2 - \alpha - \beta + \gamma, 1 - 2\gamma - n, -n; \\ 1 - 2\alpha - n, 1 - 2\beta - n\end{matrix}\right] = \frac{(2\gamma)_n (\gamma - \alpha + 1/2)_n (\gamma - \beta + 1/2)_n}{(2\alpha)_n (2\beta)_n (\gamma + 1/2)_n}$$

$$\cdot\, {}_4F_3\left[\begin{matrix}\alpha, \beta, -\gamma - n, -n; \\ \gamma, 1/2 + \alpha - \gamma - n, \frac{1}{2} + \beta - \gamma - n\end{matrix}\right].$$

Substituting $\gamma = \alpha + \beta - \frac{1}{2}$ in (4. 3), we obtain

$$(4.\,4) \quad \left[\begin{matrix}\alpha, \beta, \frac{1}{2} - \alpha - \beta - n, -n; \\ \alpha + \beta - 1/2, 1 - \alpha - n, 1 - \beta - n\end{matrix}\right] = \frac{(2\alpha)_n (2\beta)_n (\alpha + \beta)_n}{(2\alpha + 2\beta - 1)_n (\alpha)_n (\beta)_n}.$$

In (4. 1), putting $A = B = 1$, $a_1 = \alpha$, $a_2 = \beta$, $b_1 = \alpha + \beta - 1/2$, $c_1 = \alpha$, $c_2 = \beta$, $d_1 = \alpha + \beta + \frac{1}{2}$, and using (4. 4), we get a known result [2, p. 187, (3. 3)].

(*iv*) In (4. 1), putting $A = B = 1$, $a_1 = 1/2 - \alpha - \beta + \gamma$, $c_1 = 2\alpha - 1$, $c_2 = 2\beta$, $d_1 = 2\gamma - 1$, and using [1, 10. 1, (3)], we obtain

$$(4.\,5) \quad {}_3F_2\left[\begin{matrix}1/2 - \alpha - \beta + \gamma, 2 - 2\gamma - n, -n; \\ 2 - 2\alpha - n, 1 - 2\beta - n\end{matrix}\right] = \frac{(2\gamma - 1)_n (\gamma - \alpha + 1/2)_n (\gamma - \beta - 1/2)_n}{(2\alpha - 1)_n (2\beta)_n (\gamma - 1/2)_n}$$

$$\cdot\, {}_4F_3\left[\begin{matrix}\alpha, \beta, 1 - \gamma - n, -n; \\ \gamma, 1/2 + \alpha - \gamma - n, 3/2 + \beta - \gamma - n\end{matrix}\right].$$

On taking $\gamma = \alpha + \beta - \frac{1}{2}$ in (4. 5), we get

$$(4.\,6) \quad {}_4F_3\left[\begin{matrix}\alpha, \beta, \frac{3}{2} - \alpha - \beta - n, -n; \\ \alpha + \beta - 1/2, 1 - \beta - n, 2 - \alpha - n\end{matrix}\right] = \frac{(2\alpha - 1)_n (2\beta)_n (\alpha + \beta - 1)_n}{(2\alpha + 2\beta - 2)_n (\beta)_n (\alpha - 1)_n}.$$

In (4. 1), substituting $a_1=\alpha$, $a_2=\beta$, $b_1=\alpha+\beta-\frac{1}{2}$, $c_1=\alpha$, $c_2=\beta-1$, $d_1=\alpha+\beta-\frac{1}{2}$, and using (4. 6), we obtain a known result [2, p. 187, (3. 4)].

(*v*) In (4. 1), on taking $a_1=\frac{1}{2}-\alpha-\beta-\gamma$, $c_1=2\alpha$, $c_2=2\beta$, $c_3=\gamma$, $d_1=2\gamma$, $d_2=\alpha+\beta+\frac{1}{2}$, and using [1, 10. 2, (1)], we have

$$(4.7)\qquad {}_4F_3\left[\begin{matrix}\alpha,\beta,\frac{1}{2}+\alpha+\beta-2\gamma-n,-n;\\ 1/2+\alpha-\gamma-n,1/2+\beta-\gamma-n,\alpha+\beta+\frac{1}{2}\end{matrix}\right]$$

$$=\frac{(2\alpha)_n(2\beta)_n(\gamma)_n(\gamma+1/2)_n}{(2\gamma)_n(\alpha+\beta+1/2)_n\left(\frac{1}{2}+\gamma-\alpha\right)_n\left(\frac{1}{2}+\gamma-\beta\right)_n}$$

$$\cdot{}_4F_3\left[\begin{matrix}\frac{1}{2}-\alpha-\beta+\gamma,1-2\gamma-n,\frac{1}{2}-\alpha-\beta-n,-n;\\ 1-2\alpha-n,1-2\beta-n,1-\gamma-n\end{matrix}\right].$$

Putting $\gamma=\alpha+\beta-\frac{1}{2}$ in (4. 7), we get [2, p. 187, (3. 3)].

5. Particular Cases.

Here we have obtained from (3. 3) and (3. 4) many known results, by summing the series with the help of (4. 4), (4. 6) and GAUSS's theorem, etc.

(*a*) Consider (3. 3) with $A=B=1$ and $y=x$, then with

(*i*) $a_1=c-a-b$, $c_1=a$, $c_2=b$, $d_1=c$, and SAAL-SCHUTZ's theorem [8, p. 87, (29)], we get a result obtained by EULER [8, p. 60, (5)].

(*ii*) $a_1=c_1=\alpha$, $a_2=c_2=\beta$, $b_1=d_1=\frac{1}{2}+\alpha+\beta$, and [3, (10)], it yields an identity due to CLAUSEN [4, p. 185, (1)].

(*iii*) $b_1=\rho$, $d_1=\sigma$, and GAUSS's theorem, we get [4, p. 185, (2)].

(*b*) Consider (3. 4) with $A = B = 1$, $y = x$, then with

(*i*) $a_1 = c_1 = \alpha$, $a_2 = c_2 = \beta$, $b_1 = \alpha + \beta - \frac{1}{2}$, $d_1 = \alpha + \beta + \frac{1}{2}$, and (4. 4), we have an identity due to ORR [4, p. 186, (8)].

(*ii*) $a_1 = \alpha$, $c_1 = \alpha - 1$, $a_2 = c_2 = \beta$, $b_1 = d_1 = \alpha + \beta - \frac{1}{2}$, and (4. 6), it yields [4, p. 186, (9)].

(*iii*) $a_1 = \alpha$, $a_2 = \beta$, $b_1 = \alpha + \beta + 1/2$, $c_1 = \frac{1}{2} - \alpha$, $c_2 = \frac{1}{2} - \beta$, $d_1 = \frac{3}{2} - \alpha - \beta$, and [3, (11)], we get [1, p. 100].

(*c*) Consider (3. 3) with $A = -B = 1$ and $y = x$, then with

(*i*) $c_1 = a$, $d_1 = b$, and GAUSS's theorem, we get KUMMER's first formula [8, p. 125, (2)].

(*ii*) $b_1 = d_1 = \rho$, and [4, p. 104, (47)], it reduces to [4, p. 186, (3)].

(*iii*) $a_1 = c_1 = \alpha$, $a_2 = c_2 = \beta$ and DIXON's theorem [8, p. 105, (3)], we have [4, p. 186, (4)].

(*iv*) $a_1 = c_1 = \alpha$, $b_1 = d_1 = \rho$, and [8, p. 106, (4)], we get [4, p. 186, (5)].

(*v*) $a_1 = \alpha$, $b_1 = 2\alpha$, $c_1 = \beta$, $d_1 = 2\beta$, and WHIPPLE's theorem [7, p. 363, (8)], we obtain [4, p. 186, (6)].

(*vi*) $b_1 = d_1 = \rho_1$ $b_2 = d_2 = \rho_2$ and [8, p. 106, (6)], we get [4, p. 186, (7)].

(*d*) Results [4, p. 187, (12) to (15)] obtained by CHAUNDY can similarly be obtained by choosing the parameters in (3. 3) suitably.

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