Fourier series for Meijer's G-function

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1. Introduction. Recently KESARWANI [2] has given two FOURIER series for MEIJER's G-functions. Two further series of this type are given in section 3 with the help of two integrals established in section 2.

In what follows for sake of brevity a_p denotes $a_1 \cdots a_p$; δ is a positive integer and the symbol $\Delta(\delta, \alpha)$ represents the set of parameters $\frac{\alpha}{\delta}, \frac{\alpha+1}{\delta}, \cdots, \frac{\alpha+\delta-1}{\delta}$.

2. The integrals. The integrals to be established are

$$(2.1)\quad \int_0^{\pi/2} \cos(\alpha+\beta)\theta \cdot (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} \cdot G_{p,q}^{m,n}\left[z(\tan\theta)^{2\delta} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right] d\theta = \frac{(2\delta)^{\alpha+\beta-1} 2^{-\delta}}{(\pi)^{\delta-1}\Gamma(\alpha+\beta)} \cdot G_{p+2\delta,\, q+2\delta}^{m+2\delta,\, n+\delta}\left[z \,\middle|\, \begin{matrix} \Delta(\delta, 1-\alpha/2), a_p, \Delta\left(\delta, \frac{1-\alpha}{2}\right) \\ \Delta(2\delta, \beta), b_q \end{matrix}\right],$$

where

$$2(m+n) > p+q,$$
$$|\arg z| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$
$$\mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > 0.$$

$$(2.2)\quad \int_0^{\pi/2} \sin(\alpha+\beta)\theta \cdot (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} \cdot G_{p,q}^{m,n}\left[z(\tan\theta)^{2\delta} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right] d\theta = \frac{(2\delta)^{\alpha+\beta-1} 2^{-\delta}}{(\pi)^{\delta-1}\Gamma(\alpha+\beta)} \cdot G_{p+2\delta,\, q+2\delta}^{m+2\delta,\, n+\delta}\left[z \,\middle|\, \begin{matrix} \Delta\left(\delta, \frac{1-\alpha}{2}\right), a_p, \Delta(\delta, 1-\alpha/2) \\ \Delta(2\delta, \beta), b_q \end{matrix}\right],$$

where

$$2(m+n) > p+q,$$
$$|\arg z| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$
$$\mathrm{Re}(\alpha) > -1, \ \mathrm{Re}(\beta) > 0.$$

PROOF. To established (2.1), expressing the G-function as a MELLIN — BARNES type integral [1, p. 207, (1)] and interchanging the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s) z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} \cdot \int_0^{\pi/2} \cos(\alpha+\beta)\theta \cdot (\sin\theta)^{\alpha+2s\delta-1} \cdot (\cos\theta)^{\beta-2s\delta-1}\, d\theta\, ds.$$

Now evaluating the inner integral with the help of the modified form of the formula [3, p. 450, (2)], viz.

$$\int_0^{\pi/2} \cos(\alpha+\beta)\theta \cdot (\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1} d\theta = \frac{\Gamma(\pi) 2^{\alpha-1}\Gamma(\alpha/2)\Gamma(\beta)}{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha+\beta)},$$

$$\mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > 0,$$

and using the multiplication formula for the gamma function [1, p. 4, (11)], we get

$$\frac{(2\,\delta)^{\alpha+\beta-1}\,2^{-\delta}}{(\pi)^{\delta-1}\,\Gamma(\alpha+\beta)}\cdot\frac{1}{2\,\pi\,i}\int_L \frac{A}{B}\,ds\,,$$

where

$$A=\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)$$
$$\cdot\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\alpha/2+i}{\delta}+s\right)\prod_{i=0}^{2\delta-1}\Gamma\left(\frac{\beta+i}{2\,\delta}-s\right)z^s_-,$$

and

$$B=\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)$$
$$\cdot\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\frac{1-\alpha}{2}+i}{\delta}-s\right).$$

On applying [1, p. 207, (1)], the value of the integral (2. 1) is obtained.

The integral (2. 2) is established on applying the same procedure as above and using the modified form of the formula [3, p. 450, (3)].

3. The Fourier Series. The FOURIER series to be established are

$$(3.\,1)\qquad (\sin\Phi/2)^{\alpha-1}(\cos\Phi/2)^{2t-\alpha-1}$$
$$\cdot G^{m,n}_{p,q}\left[z\,(\tan\Phi/2)^{2\delta}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]=\frac{4}{(2\,\pi)^{\delta}}\sum_{r=1}^{\infty}\frac{(2\,\delta)^{2r-1}}{\Gamma(2r)}$$
$$\cdot G^{m+2\delta,n+\delta}_{p+2\delta,q+2\delta}\left[z\,\middle|\,\begin{matrix}\Delta(\delta,1-\alpha/2),a_p,\Delta\left(\delta,\frac{1-\alpha}{2}\right)\\ \Delta(2\,\delta,2\,r-\alpha),b_q\end{matrix}\right]\cos r\Phi,$$

where

$$2\,(m+n)>p+q\,,$$
$$|\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi\,,$$
$$2\,t>\mathrm{Re}\,(\alpha)>0\,,\qquad t=1\,,2\,,3\,,\cdots.$$

$$(3.\,2)\qquad (\sin\Phi/2)^{\alpha-1}(\cos\Phi/2)^{2t-\alpha-1}$$
$$\cdot G^{m,n}_{p,q}\left[z\,(\tan\Phi/2)^{2\delta}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]=\frac{4}{(2\,\pi)^{\delta}}\sum_{r=1}^{\infty}\frac{(2\,\delta)^{2r-1}}{\Gamma(2\,r)}$$
$$\cdot G^{m+2\delta,n+\delta}_{p+2\delta,q+2\delta}\left[z\,\middle|\,\begin{matrix}\Delta\left(\delta,\frac{1-\alpha}{2}\right),a_p,\Delta(\delta,1-\alpha/2)\\ \Delta(2\,\delta,2\,r-\alpha)\end{matrix}\right]\sin r\Phi,$$

where

$$2\,(m+n)>p+q\,,$$
$$|\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi\,,$$
$$2\,t>\mathrm{Re}\,(\alpha)>-1\,,\quad t=1\,,2\,,3\,,\cdots.$$

PROOF. To prove (3. 1), let

$$(3.\,3)\qquad f(\Phi)=(\sin\Phi/2)^{\alpha-1}(\cos\Phi/2)^{2t-\alpha-1}$$
$$\cdot G^{m,n}_{p,q}\left[z\,(\tan\Phi/2)^{2\delta}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]$$
$$=\sum_{r=1}^{\infty}A_r\cos r\,\Phi\,.$$

Equation (3. 3) is valid since $f(\Phi)$ is continuous and of bounded variation in the open interval $(0\,,\pi)$.

Now multiplying both sides of (3. 3) by $\cos t\,\Phi$ and integrating with respect to Φ from 0 to π, we get

$$\int_0^{\pi}\cos t\,\Phi\,(\sin\Phi/2)^{\alpha-1}(\cos\Phi/2)^{2t-\alpha-1}$$
$$\cdot G^{m,n}_{p,q}\left[z\,(\tan\Phi/2)^{2\delta}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]d\,\Phi$$
$$=\sum_{r=1}^{\infty}A_r\int_0^{\pi}\cos r\,\Phi\cos t\,\Phi\,d\,\Phi\,.$$

Using (2. 1) [with $\theta=\Phi/2\,,\ \alpha+\beta=2\,t$] and the orthogonality property of the cosine functions, we obtain

$$(3.\,4)\qquad A_t=\frac{4\,(2\,\delta)^{2t-1}}{(2\,\pi)^{\delta}\,\Gamma(2\,t)}$$
$$\cdot G^{m+2\delta,n+\delta}_{p+2\delta,q+2\delta}\left[z\,\middle|\,\begin{matrix}\Delta(\delta,1-\alpha/2),a_p,\Delta\left(\delta,\frac{1-\alpha}{2}\right)\\ \Delta(2\,\delta,2\,t-\alpha),b_q\end{matrix}\right].$$

With the help of (3. 3) and (3. 4), the result (3. 1) follows immediately.

Formula (3. 2) can be derived with the help of (2. 2) in the same manner.

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El operador elasticidad y las transformaciones adiabaticas de los gases perfectos

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Continuando con el estudio y la búsqueda de aplicaciones del operador elasticidad en diversas ramas de la Ciencia, [1], [2], presentamos en esta Nota un ejemplo sencillo de la posibilidad de aplicar el concepto de elasticidad de una función, [3], a las transformaciones adiabáticas de los gases perfectos, obtiniendo diversas expresiones para las capacidades caloríficas molares en función de las elasticidades de la presión y de la temperatura absoluta.

Cálculo de la elasticidad de la presión, respecto del volumen, en una transformación adiabática de un gas perfecto.

Diferenciando la ecuación de estado de los gases perfectos, referida a un mol, se obtiene:

$$(1) \qquad d\,T = \frac{p\,d\,V + V\,d\,p}{R}$$

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en donde R es la constante universal de los gases perfectos.

La forma diferencial del Primer Principio de Termodinámica, aplicado a un gas perfecto es, [4]:

$$(2) \qquad d\,Q = C_v\,d\,T + p\,d\,V$$

en donde C_v es la capacidad calorífica a volumen constante de un mol de gas. Por la propria definición de gas perfecto, C_v es independiente de la temperatura absoluta.

Combinando las dos expresiones anteriores se obtiene:

$$(3) \qquad d\,Q = \frac{C_v + R}{R}\,p\,d\,V + \frac{C_v}{R}\,V\,d\,p$$

En una transformación adiabática (que supondremos reversible) será:

$$(4) \qquad (C_v + R)\,p\,d\,V + C_v\,V\,d\,p = O$$

Durante la transformación adiabática, la presión es una bien conocida función del volumen, que sería facil de deducir si ello fuera necesario para nuestros fines.