dando-se a natureza desta curva em função da natureza da cónica. Estuda-se ainda as características da origem, ponto duplo da pedária, atendendo à sua posição relativamente à cónica.

RESUME

On déduit, basée sur l'équation matricielle d'une conique écrite en coordonnées cartésiennes homogènes, l'équation de sa première podaire positive par rapport à l'origine, et étudie la nature de cette courbe en fonction de la nature de la conique. On étudie aussi

les caractéristiques de l'origine, point double de la podaire, prenant en attention sa position par rapport à la conique.

SUMMARY

Based on the matricial equation of a conic in homogeneous cartesian coordinates, the equation of its first positive pedal in relation to the origin is deduced, and the nature of this curve in function of the nature of the conic studied. The characteristics of the origin (double point of the pedal) are also studied taking in account its position in relation to the conic.

Note on Jacobi endomorphisms

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1. In a recent paper [1], B. M. Put-taswamaiah studied the *Jacobi endomorphisms* of an arbitrary group G, i. e., the endomorphisms σ satisfying the condition

$$(1) \qquad ((a\,b)^\sigma\,c)^\sigma((b\,c)^\sigma\,a)^\sigma((c\,a)^\sigma\,b)^\sigma = 1$$
$$\text{for all } a, b, c \text{ in } G,$$

a^σ being the image of a under σ and 1 being the neutral element of G.

It is clear that the trivial endomorphism θ defined by $a^\theta = 1$ for every $a \in G$, is a Jacobi endomorphism. A group G which admits a non trivial Jacobi endomorphism is said to be a *Jacobi group*.

Some assertions contained in [1] are not true.

Thus, for instance, it is asserted ([1],

Lemma 1) that *a group* G *has a* (non trivial) Jacobi *endomorphism* σ *if and only if*

$$n \mid 2\,\sigma^2 + \sigma \quad \text{and} \quad a^\sigma\,b^{\sigma^2} = b^{\sigma^2}\,a^\sigma$$
$$\text{for all } a, b \text{ in } G$$

where n *denotes the exponent of* G (i. e., n is the least positive integer such that $x^n = 1$ for every $x \in G$ ([2], p. 108)).

Or this is not true, as one concludes from the following

EXAMPLE 1. Let G be the additive group of all rational numbers and let σ be the endomorphism of G defined by

$$\sigma(x) = -\frac{1}{2}\,x \quad \text{for every } x \in G.$$

Since

$$\sigma(\sigma(a+b)+c)+\sigma(\sigma(b+c)+a)+$$
$$+\sigma(\sigma(c+a)+b)=-\frac{1}{2}\left(-\frac{1}{2}(a+b)+c\right)-$$
$$-\frac{1}{2}\left(-\frac{1}{2}(b+c)+a\right)-$$
$$-\frac{1}{2}\left(-\frac{1}{2}(c+a)+b\right)=0,$$

one sees that σ is a non trivial JACOBI endomorphism (more precisely, σ is a JACOBI automorphism) and, although, one has not $n \mid 2\sigma^2 + \sigma$, since the exponent of G does not exist.

The same example shows that the assertion that *if σ is a JACOBI endomorphism of a group* G, *then* G^σ *is of odd exponent* ([1], Theorem 1, (iii)) is false.

It is also asserted that *a group has at most two JACOBI endomorphisms, one of them being the trivial endomorphism* ([1], Remarks).

This is not true, as one sees in the following

EXAMPLE 2. Let H be the direct product of G by G, where G is the group considered in Example 1. It is easy to see that the endomorphisms α, β, γ of H, defined by

$$\alpha((x,y)) = \left(-\frac{1}{2}x, -\frac{1}{2}y\right),$$

$$\beta((x,y)) = \left(-\frac{1}{2}x, 0\right),$$

$$\gamma((x,y)) = \left(0, -\frac{1}{2}y\right)$$

for every $(x,y) \in H$, are non trivial JACOBI endomorphisms.

In this note, we improve some results contained in [1]. We show that there is at most one JACOBI automorphism of a group G and such an automorphism exists, if and only if G is an abelian group with the unique square root property. We formulate a necessary and sufficient condition for a group to be a JACOBI group, in terms of a semi--direct product.

2. One says that a group G has the *square root property*, if for each $a \in G$ the equation

(2) $$x^2 = a$$

has at least one solution in G. If for each $a \in G$ the equation (2) has a *unique* solution in G, then G is said to have the *unique square root property*.

LEMMA 1. *If* G *is an abelian group having the square root property, then* G *has the unique square root property, if* (and only if) *the equation*

(3) $$x^2 = 1$$

has a unique solution in G.

PROOF. In fact, if x_1 and $x_2 \neq x_1$ are solutions of the equation (2), then 1 and $x_1 x_2^{-1} (\neq 1)$ are solutions of the equation (3).

LEMMA 2. *Let* G *be a group and let σ be an endomorphism of* G. *Then σ is a* JACOBI *endomorphism of* G, *if and only if the following conditions hold:*

(i) $a^{2\sigma^2 + \sigma} = 1$ *for every* $a \in G$;

(ii) $a^\sigma b^{\sigma^2} = b^{\sigma^2} a^\sigma$ *for all* a, b *in* G.

(If α and β are mappings of G into G, then $a^{\alpha\beta}$ means $(a^\alpha)^\beta$ and $a^{\alpha+\beta}$ means $a^\alpha a^\beta$).

PROOF. Indeed, if σ is a JACOBI endomorphism of G, then from (1) it results

$$(4) \qquad a^{\sigma^1} b^{\sigma^1} c^\sigma b^{\sigma^1} c^{\sigma^1} a^\sigma c^{\sigma^1} a^{\sigma^1} b^\sigma = 1 \,,$$

for all a, b, c in G.

By setting $b = c = 1$ in (4), one obtains

$$a^{\sigma^1} a^\sigma a^{\sigma^1} = 1 \quad \text{for every} \quad a \in G$$

and, consequently, (i) holds.

By setting $c = 1$ in (4), one obtains

$$a^{\sigma^1} b^{\sigma^1} b^{\sigma^1} a^\sigma a^{\sigma^1} b^\sigma = 1 \quad \text{for all} \quad a, b \text{ in } G$$

and, since one has $b^{2\sigma^1} = b^{-\sigma}$ and $a^{\sigma + \sigma^1} = a^{-\sigma^1}$ by (i), it results

$$a^{\sigma^1} b^{-\sigma} a^{-\sigma^1} b^\sigma = 1 \quad \text{for all} \quad a, b \text{ in } G \,,$$

which is equivalent to condition (ii).

Conversely, if σ satisfies the conditions (i) and (ii), then one has

$$a^{\sigma^1} b^{\sigma^1} c^\sigma b^{\sigma^1} c^{\sigma^1} a^\sigma c^{\sigma^1} a^{\sigma^1} b^\sigma$$
$$= a^{\sigma^1} b^{\sigma^1} b^{\sigma^1} (c^\sigma c^{\sigma^1} c^{\sigma^1}) a^\sigma a^{\sigma^1} b^\sigma, \text{ by } (ii)$$
$$= a^{\sigma^1} b^{\sigma^1} b^{\sigma^1} a^\sigma a^{\sigma^1} b^\sigma, \text{ by } (ii) \text{ and } (i)$$
$$= a^{\sigma^1} a^\sigma (b^{\sigma^1} b^{\sigma^1} b^\sigma) a^{\sigma^1}, \text{ by } (ii)$$
$$= 1, \text{ by } (i) \text{ and } (ii),$$

proving that σ is a JACOBI endomorphism.

THEOREM 1. *If σ is a JACOBI endomorphism of the group* G, *then* G^σ *is an abelian group having the unique square root property.*

PROOF. Indeed, let $x, y \in G^\sigma$, i. e., one has $x = z^\sigma$ and $y = t^\sigma$ for suitable elements $z, t \in G$. Then, by repeated use of (i) and (ii), one obtains

$$xy = z^\sigma t^\sigma = z^\sigma (t^{-1})^{2\sigma^1} = z^\sigma (t^{-1})^{\sigma^1} (t^{-1})^{\sigma^1}$$
$$= (t^{-1})^{\sigma^1} z^\sigma (t^{-1})^{\sigma^1} = (t^{-1})^{\sigma^1} (t^{-1})^{\sigma^1} z^\sigma$$
$$= (t^{-1})^{2\sigma^1} z^\sigma = t^\sigma z^\sigma = yx \,,$$

for all $x, y \in G^\sigma$, which proves that G^σ is an abelian group.

Now, let a be any element of G^σ, i. e., $a = c^\sigma$ for a suitable element $c \in G$. Since, by condition (i) of Lemma 2, one has

$$a = c^\sigma = c^{-2\sigma^1} = (c^{-\sigma^1})^2 \quad \text{and} \quad c^{-\sigma} = (c^{-\sigma})^\sigma,$$

one sees that the equation $x^2 = a$ has at least one solution in G^σ, namely, $x = c^{-\sigma^1}$.

Let us show that the equation $y^2 = 1$ has only one solution in G^σ.

In fact, it is easy to see that the restriction of σ to G^σ is an automorphism.

One has clearly $G^{\sigma^1} \subseteq G^\sigma$, since $x^{\sigma^1} = (x^\sigma)^\sigma$ for every $x \in G$. On the other hand, if $x = z^\sigma \in G^\sigma$, then, since $z^\sigma = (z^{-2})^{\sigma^1}$ by condition (i) of Lemma 2, one concludes that $x \in G^{\sigma^1}$, i. e., $G^\sigma \subseteq G^{\sigma^1}$, and thus the restriction of σ to G^σ is a surjective endomorphism.

Moreover, if $x = z^\sigma \in G^\sigma$ is such that $x^\sigma = 1$, then one has obviously

$$1 = z^{\sigma^1} = z^{-\sigma^1 - \sigma} \,,$$

hence

$$x = z^\sigma = z^{-\sigma^1} = (z^{\sigma^1})^{-1} = 1 \,,$$

meaning that the restriction of σ to G^σ is also an injective endomorphism and, consequently, it is an automorphism.

From this it follows that the equation $y^2 = 1$ has only the solution $y = 1$ in G^σ, because

$$y^2 = 1 \text{ implies } 1 = (y^2)^{-\sigma^1} = y^{-2\sigma^1} = y^\sigma$$

and hence one concludes, by Lemma 1, that G^σ has the unique square root property, as it was to be proved.

THEOREM 2. *If G is an abelian group with the unique square root property, then there exists exactly one JACOBI automorphism of G.*

PROOF. Indeed, let $\rho : G \to G$ be the mapping defined by the following condition:

For each $a \in G$, a^ρ is the (unique) solution of the equation $x^2 = a$ *in* G.

Since G is an abelian group, from the equations

$$x^2 = a \text{ and } y^2 = b, \text{ with } a, b \text{ in } G,$$

it follows $(xy)^2 = x^2 y^2 = ab$, that is to say,

$$(ab)^\rho = xy = a^\rho b^\rho, \text{ for all } a, b \text{ in } G,$$

and thus ρ is an endomorphism of G.

Furthermore, for every element $a \in G$, one has $(a^2)^\rho = (a^\rho)^2 = a$ and, since

$$a^\rho = b^\rho \text{ implies } a = (a^\rho)^2 = (b^\rho)^2 = b,$$

one concludes that ρ is an automorphism of G.

Now, by setting

$$x^\sigma = (x^{-1})^\rho \text{ for every } x \in G,$$

one sees that σ is an automorphism of G.

Since G is an abelian group, the condition (ii) of Lemma 2 is trivially satisfied. Moreover, one has

$$x^{2\sigma^2 + \sigma} = ((x^\sigma)^\sigma)^2 x^\sigma = ((x^{-\rho})^{-\rho})^2 x^{-\rho}$$
$$= (x^{\rho^2})^2 x^{-\rho} = ((x^2)^\rho)^\rho x^{-\rho}$$
$$= x^\rho x^{-\rho} = 1,$$

which completes the proof that σ is a JACOBI automorphism of G.

Let us suppose that τ is also a JACOBI automorphism of G.

Then, for each $x \in G$, the element $y = (x^{-1})^\tau$ is a square root of x.

In fact, from $x^\tau y = 1$, it follows

$$1 = x^{2\tau} y^{2\tau} = x^{-\tau} (y^2)^\tau = (x^{-1} y^2)^\tau$$

and since τ is an automorphism, one has $x^{-1} y^2 = 1$, i. e., $y^2 = x$.

In view of the fact that G has the unique square root property, one has $(x^{-1})^\sigma = (x^{-1})^\tau$ for every $x \in G$ and hence $\sigma = \tau$, as wanted.

From Theorem 1 and 2, it results immediately the following

THEOREM 3. *If* G *is a group, then there is at most one* JACOBI *automorphism of* G *and such an automorphism exists, if and only if* G *is an abelian group satisfying the unique square root condition.*

In particular, one can state the following

COROLLARY. *If* G *is an abelian group of odd exponent, then there exists exactly one* JACOBI *automorphism of* G.

In fact, if $2n + 1$ is the exponent of G, from $x^{2n+1} = 1$ for every $x \in G$, it results

$$(x^{-n})^2 = x \text{ for every } x \in G,$$

i. e., G has the square root property.

Since $y^2 = 1$ implies

$$y = y \cdot 1 = y \cdot y^{2n} = y^{2n+1} = 1,$$

one concludes by Lemma 1 that G has the unique square root property and, by Theorem 3, that there is only one JACOBI automorfism σ of G, defined by $x^\sigma = x^n$ for every $x \in G$.

The Corollary to Theorem 3 improves Lemma 2 in [1].

REMARK. In [1], Theorem 3, it is proved that G *is a* JACOBI *group under an inner automorfism* σ *of* G, *then* G *is a direct product of cyclic groups each of order 3.*

This is true and it may be added that σ is the identity automorphism, as one concludes immediately by means of the Corollary above.

3. Let N be a normal subgroup of the group G and let H be a subgroup of G. One says that G is the *semi-direct product* of N by H, if for each $x \in G$ there is exactly one element a in H such that $x N = a N$.

THEOREM 4. *A group* G *is a* JACOBI *group, if and only if* G *is the semi-direct product of a proper normal subgroup by an abelian subgroup with the unique square root property.*

PROOF. Let us suppose that G is a JACOBI group and let σ be a non trivial JACOBI endomorphism of G. It is known that C^σ is isomorphic to the quotient group G/N, where N denotes the kernel of σ.
Since

$$(x \cdot x^{2\sigma})^\sigma = x^{2\sigma^2 + \sigma} = 1 \text{ for every } x \in G,$$

one has $x \cdot x^{2\sigma} \in N$ and, therefore, $x N = x^{-2\sigma} N$ with $x^{-2\sigma} \in G^\sigma$.
Moreover, if the element $y^\sigma \in G^\sigma$ satisfies the condition $x N = y^\sigma N$, then one has $x y^{-\sigma} \in N$, i. e., $(x y^{-\sigma})^\sigma = 1$, hence $x^\sigma y^{-\sigma^2} = 1$.
From this it follows

$$1 = x^{2\sigma} y^{-2\sigma^2} = x^{2\sigma} y^\sigma,$$

i. e., $y^\sigma = x^{-2\sigma}$.
This means that G is the semi-direct product of N by G^σ, because N is a proper normal subgroup of G and G^σ is a subgroup of G. By Theorem 1, we know that G^σ is an abelian group with the unique square root property.
Conversely, let us suppose that G is the semi-direct product of a proper normal subgroup N by an abelian subgroup H having the unique square root property. Then the groups H and G/N are isomorphic.
Let ν be the natural homomorphism of G onto G/N and let μ be the isomorphism of G/N onto H defined by the condition

$$(a N)^\mu = a'.$$

where a' is the (unique) element of H such that $a N = a' N$.
Finally, let τ be the (unique) JACOBI automorphism of H (Theorem 2). By setting

$$\sigma = \nu \mu \tau,$$

one gets clearly an endomorphism of G and one has $G^\sigma = H$.
It is immediate that *the kernel of* σ *is* N.
In fact, if $x \in N$, then $x^\nu = 1$, hence $(x^\nu)^\mu = 1$ and consequently,

$$x^\sigma = 1^\tau = 1.$$

Conversely, if $y^\sigma = 1$, then one has $x'^\tau = 1$ and, since τ is an automorphism of H, it follows $x' = 1$, that is to say, $x N = N$ and therefore, $x \in N$.
In view of the fact that x'^τ is the (unique) square root of x'^{-1} in H, one has $x^{2\sigma} = x'^{-1}$, that is to say, $x N = x^{-2\sigma} N$ and so $x^{2\sigma} \cdot x \in N$.
From this it follows

$$x^{2\sigma^2 + \sigma} = (x^{2\sigma} \cdot x)^\sigma = 1 \text{ for every } x \in G.$$

Moreover, one has clearly

$$x^\sigma y^{\sigma^2} = y^{\sigma^2} x^\sigma \text{ for all } x, y \text{ in } G,$$

since the group $G^\sigma (= H)$ is an abelian group.
Thus, the conditions (i) and (ii) of Lemma 2 are satisfied by the endomorphism σ.
This means that σ is a JACOBI endomorphism and, since this endomorpism is non trivial because $N \neq G$, one concludes that G is a JACOBI group, as wanted.
This theorem improves Theorem 3 in [1].

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