os valores

$$p_3(0)=\frac{55}{216}, p_3(1)=\frac{55}{216}, p_3(2)=$$
$$=\frac{53}{216}, p_3(3)=\frac{53}{216}, \text{ com } \sum p_3(\nu)=1.$$

Como tínhamos afirmado de início, os valores de $p_n(\nu)$ tendem alternadamente para o limite $1/4$, independente de ν. Isso é agora evidente a partir de (10), pois que por $(-\sqrt{2}/6)^n \to 0$, ao passarmos ao limite, a segunda parcela do lado direito de (10) tende para o vector zero. Este resultado é, aliás, susceptível de uma justificação heurística, pois, sendo o dado ideal (como considerámos na formulação do problema), no limite para um número infinito de lançamentos obtém-se uma simetria para as 4 diferentes congruências possíveis, pelo que se deverá ter $p_n(\nu) \to 1/4$, para qualquer ν (este raciocínio é, porém, falso no caso de o dado ser viciado, e. g. se só forem positivas as probabilidades de obtenção de faces com um número par de pontos). Teríamos chegado ainda ao mesmo resultado, observando que a matriz A define uma aplicação de contracção no sub-espaço do vulgar espaço métrico euclideano a 4 dimensões formado pelos possíveis vectores p_n (de coordenadas $0 \leq p_n(\nu) \leq 1$, com $\Sigma p_n(\nu)=1$). Pelo teorema de BANACH relativo aos pontos fixos, deverá haver um e um só ponto fixo, que logo se vê ser o vector cujas 4 coordenadas são todas iguais a 1/4.

NOTA. A sucessão $\{p_n; n=0,1,2,\cdots\}$ dos vectores p_n constitui um dos exemplos mais simples de uma sucessão de distribuições de probabilidade em que cada uma delas depende apenas da precedente p_{n-1}, tal como num processo iterativo simples $\{p_n; n= =0,1,\cdots\}$ é portanto um exemplo, extremamente simples, de uma cadeia de MARKOV.

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Pari-mutuel betting

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Pari-mutuel betting is a form of betting on horses in which those who bet on the winning horse share the total amount bet on all horses less a small per cent which is paid to the management. Considering that statisticians and mathematicians have always been particularly interested in gambling systems, it is surprising that pari-mutuel betting has received such sparse attention. This lack of attention is perhaps attributable to the inherent difficulty of determining the actual probability that a horse will win a race, since this probability depends on the other horses in the race, the conditions of the track, and a plethora of other factors. To circumvent this difficulty, let us suppose that the better has a scheme for assigning probabilities of winning to the various horses. Actually, this is not an unreasonable assumption. The various race tracks publish what is called the «Morning Line», which lists the track handicapper's estimates of the odds on all horses. Thus, if the track percentage, r, is ignored, the usual odds-probability relationship holds:

$$Odds = \frac{1-p}{p} \text{ or } p = \frac{1}{Odds+1}.$$

Other natural choices of probability distribution might be based on the proportion of the total pool bet on each horse, or schemes weighting the «picks» of various handicappers.

Suppose we have a race of n horses. Let h_i designate the event that the ith horse wins the race, and let $P=(p_1, \cdots, p_n)$ denote a probability distribution over the events $h_1, \cdots, h_n$ where $p_i \geq 0$ for $i=1, \cdots, n$ and $\sum_i p_i = 1$. Let us assume that the better is the last person to bet and can bet an amount α. (We will later assume that $\alpha = 1$.) Further, let us designate by A_i the total amount already bet on the event h_i, and let $A = \sum_{i=1}^{n} A_i$. Finally, let r designate the proportion of the total amount bet kept by the track. Let us first treat the problem classically and see why this approach is not fruitful. (I will follow the analysis made by BOREL [1].) Suppose the better has placed an amount α on the event h_i. Assuming that $r=0$, we see that if h_i occurs he receives $\alpha \sum_{j \neq i} A_j/(A_i+\alpha)$, and if h_i does not occur he loses α. His expected gain is then

$$x(\alpha) = \frac{\alpha \sum_{j\neq i} A_j}{A_i+\alpha} p_i - (1-p_i)\alpha .$$

This function vanishes at $\alpha = 0$ (i. e., no bet, no gain) and also at $\alpha_0 = (p_i A - A_i)/ /(1-p_i)$, and is positive on $(0, \alpha_0)$ as long as $\alpha_0 > 0$ (which is equivalent to $p_i A - A_i > 0$, or $A_i/p_i < A$). To maximize $x(\alpha)$, set $\frac{dx(\alpha)}{d\alpha} = 0$ and obtain

$$(A_i+\alpha) = A_i(1+\alpha_0/A_i)^{\frac{1}{2}} .$$

If α_0/A_i is small, then $(1+\alpha_0/A_i)^{\frac{1}{2}} \doteq \doteq 1+\alpha_0/2A_i$ so that $\alpha \doteq \alpha_0/2 = \frac{(p_i A - A_i)}{2(1-p_i)}$.

Thus, if we can determine a probability distribution P on $h_i, \cdots, h_n$, and know $A_1, \cdots, A_n$ we know how much to bet. The condition that there exist an h_i such that $A_i/p_i < A$ always holds unless $A_i p_j = A_j p_i$ for all i, j, since, if not, then $A_i/p_i \geq A$ for all i. Suppose that strict inequality held for one of the A_i's, then

$$\sum_{i=1}^{n} A_i > \sum_{i=1}^{n} p_i A = A \text{ and then } A > A ,$$

which is a contradiction. The entire discussion so far hinges on the choice of the subjective probability distribution P, and therein lies the weakness of the analysis. Actually, this subjectivity is the motivating factor for betting, since each better can choose his own P and under it bet so as to have a positive expectation. But this subjectivity raises a serious difficulty. Each better can view himself as the last better and bet according to his P and his assessment of what the final A_i's will be. But his bet affects the A_i's, thus we must answer the question of whether there exist final A_i's and individual bets which are compatible with both the various betters' strategies and the pari-mutuel system.

This question is discussed in a paper by EDMUND EISENBERG and DAVID GALE [2].

Suppose we have m individual betters G_1, $G_2, \ldots, G_m$, and n events $h_1, \ldots, h_n$. Let $\overline{P}=$ «subjective probability» matrix whose i, jth element is p_{ij}, the subjective probability that better G_i assigns to the event h_j. Suppose that G_i has a fixed budget B_i, and assume that he bets b_{ij} on h_j according to the following strategy: He will bet his B_i according to some partition $(b_{i1}, b_{i2}, \ldots, b_{in})$ on those events h_j whose p_{ij}/π_j is maximizd, where $\pi_j = A_j/A$ in our old notation (i. e., he bets on those horses where his opinion diverges most drastically from the consensus).

Choose a unit of money so that $\sum_{i=1}^{m} B_i = 1$.

Further assume each column of $\overline{P}$ contains at least one positive element (otherwise no one would bet on the horse corresponding to the column of zeros).

The pari-mutuel system requires that:

(a) $\sum_{j=1}^{n} b_{ij} = B_i$,

(b) $\sum_{i=1}^{m} b_{ij} = \pi_j$ (since $\sum_{i=1}^{m} B_i = 1$),

(c) if $\mu_i = \max_s \; p_{is}/\pi_s$, and $b_{ij} > 0$, then $p_i = p_{ij}/\pi_j$, where $s = 1, \cdots, n$;

(c) simply states that each gambler is betting on those horses for which his expectation is a maximum. (This is easily seen since the return — neglecting track percentage — is $\frac{1-\pi_j}{\pi_j}$, for each unit bet; so that $x_i(1) = \sum_j p_{ij}(1/\pi_j - 1)\, b_{ij} = \sum_j p_{ij}\, b_{ij}/\pi_j - \sum_j p_{ij} b_{ij}$. Thus, for example, expectation is positive if G_i bets on those horses where $p_{ij}/\pi_j > 1$, or just on the horse with the maximum ratio.) Any set of π_j's and b_{ij}'s satisfying (a), (b), and (c) are called equilibrium probabilities and bets. Before stating the basic theorem, let us define a function φ with mn arguments ξ_{ij} as:

$$\varphi(\xi_{11}, \cdots, \xi_{mn}) = \sum_{i=1}^{m} B_i \ln\Big(\sum_{j=1}^{n} p_{ij}\xi_{ij}\Big)$$

where $\xi_{ij} \geq 0$ for all i, j and $\sum_{i=1}^{m} \xi_{ij} = 1$.

Note that φ is continuous on its domain of definition and the domain is compact. Thus there exists a max value of φ at some point, say $(\overline{\xi}_{11}, \cdots, \overline{\xi}_{mn})$.

THEOREM. *A set of equilibrium probabilities* π_j *and bets* b_{ij} *are given by*

$$\max_i \frac{\partial \varphi}{\partial \overline{\xi}_{ij}} = \pi_j = \max_i \frac{B_i p_{ij}}{\sum_s p_{is}\overline{\xi}_{is}}\, b_{ij} = \overline{\xi}_{ij}\, \pi_j .$$

PROOF. These must satisfy (a), (b) and (c).

(b) $\sum_{i=1}^{m} b_{ij} = \pi_j \sum_{i=1}^{m} \overline{\xi}_{ij} = \pi_j$ by the restriction on the domain of definition.

Before we prove (a) and (c), note that if $\xi_{ij} > 0$, then $\pi_j = \frac{\partial \varphi}{\partial \overline{\xi}_{ij}}$ since if this were not the case, then there exists a k such that $\pi_j = \frac{\partial \varphi}{\partial \overline{\xi}_{kj}} > \frac{\partial \varphi}{\partial \overline{\xi}_{ij}}$. We can now perturb φ by a small decrease ∂ in $\overline{\xi}_{ij}$ and a ∂ increase in $\overline{\xi}_{kj}$, to obtain a greater value. But since we are already at a maximum, we have arrived at a contradiction.

Accordingly, we now know that $b_{ij} = \overline{\xi}_{ij}\pi_j = \overline{\xi}_{ij} \frac{B_i p_{ij}}{\sum_s p_{is}\overline{\xi}_{is}}$, which implies that

$$\sum_j b_{ij} = B_i \frac{\sum_j \overline{\xi}_{ij} p_{ij}}{\sum_s p_{is}\overline{\xi}_{is}} = B_i, \text{ verifying } (a).$$

Finally, to show (c), note that since none of the columns of the matrix $\overline{P}$ are identically zero, each π_j is greater than zero. Thus $p_{il}/\pi_l \leq \frac{\sum_s p_{is}\overline{\xi}_{is}}{B_i}$ for all l, with equality at $l=j$ (from the definition of π_l in the Theorem). Since $b_{ij}>0$ just when $\overline{\xi}_{ij}>0$, then $\pi_j = \frac{B_i p_{ij}}{\sum_s p_{is}\overline{\xi}_{is}}$ and thus $\mu_i = \max_s p_{is}/\pi_s = p_{ij}/\pi_j$, so (c) is satisfied. Q. E. D.

Accordingly, we have shown that there exist final track probabilities (the π_j's) and individual bets (the b_{ij}'s) compatible with both the pari-mutuel system and individual betting strategies.

Let us now attempt to rid ourselves completely of the problem of estimating P by viewing the pari-mutuel system as a two-person game (the better vs. nature) and attempt to find a minimax solution. Suppose we denote by $s(\alpha_1, \ldots, \alpha_n)$ the betting strategy which bets the amount α_i on the event h_i where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$ (in terms of some unit). If h_i occurs, then the pay off to the better is:

$$-\alpha_i\left(\frac{(1-r)(A+1)-(A_i+\alpha_i)}{A_i+\alpha_i}\right)+\sum_{j\neq 1}\alpha_j = 1-\frac{(1-r)(A+1)}{A_i+\alpha_i}\alpha_i .$$

(Note that we are now designating a win by the better as negative quantity.)

Suppose that the probability of h_i is p_i. Then the expected loss for strategy s relative to P is

$$L(P,s)=\sum_{i=1}^{n} P_i\left(1-\frac{(1-r)\ (A+1)\alpha_i}{A_i+\alpha_i}\right) = 1-(1-r)(A+1)\left[1-\sum_{i=1}^{n}\frac{A_i p_i}{A_i+\alpha_i}\right]$$

Thus we can minimize $L(P,s)$ by minimizing $\sum_{i=1}^{n}\frac{A_i p_i}{A_i+\alpha_i}$ subject to the conditions that $\sum_i \alpha_i = 1$, and $\alpha_i \geq 0$ for all i. Choose a μ such that $p_\mu > 0$; then

$$\frac{\partial L(P,s)}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i}\left(\frac{A_i p_i}{A_i+\alpha_i}+\frac{A_\mu p_\mu}{\left(A_\mu+1-\sum_{j\neq i}\alpha_j\right)}\right)=0$$

yielding $\frac{A_i p_i}{(A_i+\alpha_i)^2}=\frac{A_\mu p_\mu}{(A_\mu+\alpha_\mu)^2}$ for those i where $p_i>0$, which gives

$$\alpha_i=\frac{\sqrt{A_i p_i}\,(A_\mu+\alpha_\mu)}{\sqrt{A_\mu p_\mu}}-A_i .$$

Noting that

$$1-\alpha_\mu=\sum_{\substack{i\neq\mu\\ p_i>0}}\frac{\sqrt{A_i p_i}\,(A_\mu+\alpha_\mu)}{\sqrt{A_\mu p_\mu}}-\sum_{i\neq\mu}A_i ,$$

we have $\alpha_\mu=\frac{\sqrt{A_\mu p_\mu}\,(1+A_0)}{\sum_i\sqrt{A_i p_i}}$ where

$$A_0=\sum_{\substack{i\\ p_i>0}}A_i .$$

Thus by substitution,

$$\alpha_i=\frac{\sqrt{A_i p_i}\,(1+A_0)}{\sum_i\sqrt{A_i p_i}}-A_i \text{ and}$$

$$L(P,s)= =1-(1-r)(A+1)\left[1-\frac{(\sum_i\sqrt{A_i p_i})^2}{1+A_0}\right].$$

Now suppose that the p_i's are proportional to the track returns $A_1, \cdots, A_n$, say

$p_i = A_i/A$. Then $A_0 = A$, $\alpha_i = A_i/A$ and

$$L(P, s) = 1 - (1-r)(A + 1 - A^2/A) = \\ = 1 - (1-r) = r.$$

This is obviously BAYES against $P = (A_1/A, A_2/A, \cdots, A_n/A)$ and since it has constant risk, it is a mínimax solution. Thus betting according to this strategy, the better cannot lose more than 100 r per cent of his unit bet, no matter what the real probability distribution is on $h_1, \cdots, h_n$. This system is easy to use in any real gambling situation since one does not have to bother estimating P, but like all minimax strategies it is quite pessimistic. Thus, if the better could actually estimate P with some sucess, it would be to his advantage to go through the more complicated mathematics of the general solution.

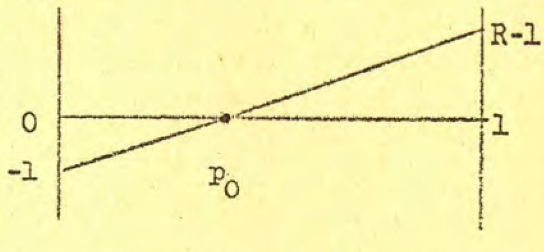
Before leaving this discussion, let us note that if α_i is small w. r. t. A_i (as it would probably be), then

$$L(P, s) \doteq 1 - (1-r)(A+1) \sum_{i=1}^{n} \frac{\alpha_i p_i}{A_i},$$

which is maximized by taking p_i/A_i largest and setting $\alpha_i = 1$. This is equivalent to choosing A_i/p_i smallest, which is in effect what BOREL's method told us to do.

Another approach to the pari-mutuel betting problem has been given by R. CLAY SPROWL in 1950. It is a relatively crude method, but the approach is quite different from either one that we have considered so far.

SPROWL leaves aside the question of how one should bet, and considers only the question «When?» He assumes that the better has a system which works with probability p (p being unknown). If he assumes that the payoff is $(R-1)$ if he wins on a unit bet (here $(R-1)$ corresponds to the odds; or $1 - \frac{(1-r)(A+1)}{A_i+1}$ in our old notation), his expectation is $x(p) = (R-1)p - (1-p)$. Now if we plot this as a function of p, we get a straight line which intersects the p-axis at the point p_0.



Now p_0 is a function of R, and this is the cornerstone of his paper. Given R, determine $p_0 = 1/R$. If $p > p_0$, bet; otherwise do not. The better, however, must determine what p is and the rest of the paper is devoted to a decision-theoretic derivation of the natural estimate (number of previous wins)/(number of races bet). He ends his discussion with the first real experiment published in the field. He uses two systems; the first being bet the favorite, the second being bet the favorite if $\hat{p} > p_0$, where $\hat{p}$ is the proportion of wins in the previous n races if he had bet the favorite. The results are:

	Sistem I	Sistem II
Races Bet	75	23
Races Won	20	4
Total Bet	$150.00	$46.00
Total Won	$128.50	$50.60
Profit	$-21.50	$ 4.60

Interpretation of the results are left to the reader.

The most recent article on the subject is by RICHARD N. ROSETT. It was published in 1965 in, of all places, *The Journal of Political Economy*. The motivation behind his research is the query as to whether gamblers are rational. He formulates a rationality hypothesis as follows.

If a gambler must choose between two single return bets in which he risks losing one unit, he will: (*a*) if the probabilities of winning are equal, choose that bet with the greatest return; (*b*) if the returns are equal, choose that with the greatest probability; (c) always choose an event which in both return and probability of occurrence are greater than in the other.

Three systems of betting are discussed in the paper; the first is called the Martingale and works like this: Select n horses each running in a different race, and suppose they pay $R_1, \ldots, R_n$ in returns, and that there exist some actual probabilities $p_1, \ldots p_n$ of their winning (here subscripts denote the order of the running of the race). Split your bet into sums of money $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i > 0$, and $\alpha_i (R_i - 1) - \sum_{j=1}^{i-1} \alpha_j = B$ for $i = 1, \ldots, n$. Solve this system of equations for the α_i's and bet α_i on your choice in the ith race, but do this sequentially until either all n races have been bet, or a race is won. If you lose all n races, you lose one unit with probability $\prod_{i=1}^{n} (1-p_i)$. If one of the horses wins (with probability $1 - \prod_{i=1}^{n} (1-p_i)$), you win B. This method combines low-probability, high return bets, into one system with high probability and low return.

A parallel system for a single race can also be constructed. (Sprowl calls it the combination.) Again, split your bet into amounts $\alpha_1, \ldots, \alpha_k$ (note that k is not necessarily equal to the total number of horses running in the race), but according to the system of equations: $\alpha_i (R_i - 1) = B$ $i = 1 \ldots, k$ and $\sum_{i=1}^{k} \alpha_i = 1$. Bet on all k horses. If p_i is the proability of h_i, the better wins B with probability $\sum_{i=1}^{k} p_i$, and loses one with probability $1 - \sum_{i=1}^{k} p_i$. This is again a high probability, low return bet.

The final system is called the parlay and consists of choosing n horses in n races and betting 1 unit on the first choice in the first race. If you win, bet R_1 on the choice in the second race and continue until you lose a race or win all n. If you win, the return is $\prod_{i=1}^{n} R_i - 1$ with probability $\prod_{i=1}^{n} p_i$, and the loss is 1 (although much greater from a Regret point of view if one were to win the first $n-1$ races) with probability $1 - \prod_{i=1}^{n} p_i$. This is a high-return low probability bet. It must be emphasized, that all the choices in the above three systems are made at one time, so in effect they comprise a single system or a single bet.

Now, assuming that the betters form a market (this is similar to the perspective of Eisenberg and Gale), the rationality hypothesis (which is equivalent to a functional relationship between R and p which is monotone decreasing in p), and the possibility of combining bets through parlays and Martingales, place certain constraints on the function relating return to probability of winning, which are:

Given any probability of winning p^* and the return associated with it R^*, then

$$p^r \leq R(p) \leq [1-(1-p)^c]^{-1} \qquad \text{if } 0 \leq p \leq p^*$$
$$[1-(1-p)^c]^{-1} \leq R(p) \leq p^r \qquad \text{if } p^* \leq p \leq 1$$

where R is the return associated with p at *equilibrium* and *subject* to the rationality hypothesis, $r = \ln R^* / \ln p^*$, and

$$c = \ln(1 - 1/R^*) / \ln(1 - p^*).$$

Let us give some indication of how these limits are derived. Suppose $p^* \leqq p \leqq 1$ and $R(p) > p^r$. Choose such a p^*, then there exists a q such that $pq = p^*$ with $q > p^*$, and then $\log_{p^*} p + \log_{p^*} q = 1$. Now take a bet with probability p and another with probability q and form a parlay. The return is $R(p)R(q)$ with probability $pq = p^*$, but since $p, q > p^*$, $R(p)R(q) > R^{*\ln p/\ln p^*}$. $R^{*\ln q/\ln p^*} = R^*$. Thus we have a bet $(p^*, R(pq))$ which has for the same probability, a greater return. Hence, (p^*, R^*) would never be chosen, which is a contradiction since this is the equilibrium relationship. Similarly, suppose that $R(p) <$ $< [1-(1-p)^c]^{-1} = \left[1 - 1/R^*)^{\frac{\ln(1-p)}{\ln(1-p^*)}}\right]^{-1}$. Form a Martingale from n bets like (p^*, R^*) such that $p = 1-(1-p^*)^n$. From the rationality hypothesis, $R(p) - R(1-(1-p^*)^n)$. But the return from the Martingale is $[1-(1-1/R^*)^n]^{-1} = [1-(1-1/R^*)^c]^{-1}$ since $c = \dfrac{\ln(1-p)}{\ln(1-p^*)} = \dfrac{n\ln(1-p^*)}{\ln(1-p^*)} = n$ by construction. This Martingale has the same probability of winning as the initial bet, but the return is greater. Thus according to the rationality hypothesis, the first bet would never be chosen which is a contradiction. Similar manipulations will verify the inequalities for $0 \leqq p \leqq p^*$.

Now $R(p) = p^r$ is the *only* function relating returns to probability for which it is true that for every bet on a single horse any parlay having the same probability will pay the same return. Similarly $R(p) =$ $= [1-(1-p)^c]^{-1}$ is the *only* function relating returns to probability, for which it is true that for every bet on a single horse, any Martingale having the same probability will pay the same return.

Let us verify this assertion. Select bets with probabilities $p_1, \cdots, p_n$ such that $\prod_{i=1}^{n} p_i = p_0$, then if the above relation holds, the return is $R_0 = \left(\prod_{i=1}^{n} R(p_i)\right) = \left(\prod_{i=1}^{n} p_i^r\right) =$ $= \left(\prod_{i=1}^{n} p_i\right)^r = p_0^r$.

Now suppose that we have any other function $R = f(p)$ which has the properties:

(*a*) $f\left(\prod_{i=1}^{n} p_i\right) = \prod_{i=1}^{n} f(p_i)$,

(*b*) f is monotone decreasing over $(0, 1)$ (note: (*a*) assures us of the proper parlay relationship, (*b*) is necessitated by the rationality hypothesis), and

(*c*) f does not have the form $f(p) = p^r$,

choose two values p_1, p_2 and find $R_1 = f(p)$, $R_2 = f(p_2)$ so that $f(p_1) = p_1^{r_1}$ $f(p_2) = p_2^{r}$ where $r_1 > r_2$ (note that $r_2 < r_1 < 0$). Thus for all p in $(0, 1)$, $p^{r_1} < p^{r_2}$.

Since (p_1, R_1) and (p_2, R_2) both satisfy (*a*), so do all points of form (p_1^n, R_1^n), (p_s^m, R_s^m) where n and m are positive integers. Suppose we found a point $(p_2^{m_0}, R_2^{m_0})$ such that $p_2^{m_0} > p_1^{m_0}$ for some m_0 and n_0. Then since $p_2^{m_0} > p_1^{n_0}$ implies $p_2^{m_0 r_2} > p_1^{n_0 r_2}$ and $r_1 > r_2$ implies $p_1^{n_0 r_2} > p_1^{n_0 r_1}$ then $R_2^{m_0} =$ $= p_2^{m_0 r_2} > p_1^{n_0 r_1} = R_1^{n_0}$ and we would violate the rationality (and thus the monotonicity) property. To see that this always occurs look at the line $R = R_1^{n+1}$ (i. e., $R_1(p_1^{n+1})$); it intersects the curve $R_2 = p^{r_2}$ at the point $p_2^* = (R_1^{n+1})^{1/r_2} = (p_1^{n+1})^{r_1/r_2}$. Now $(p_1^{n+1})^{r_1/r_2} > p_1^n$ if and only if $n > \dfrac{r_1/r_2}{1 - r_1/r_2}$ (remember that both r_1 and r_2 are negative, so $r_1/r_2 < 1$). Thus we have a point $(p_2^*, R(p_2^*))$ such that $p_1^* > p_1^n$, which gives us the violation of the rationality hypothesis. Thus $R(p) = p^r$ is the unique functional form satisfying (a) and (b).

Let us now show that $R(p)=[1-(1-p)^c]^{-1}$ is the unique function satisfying the requirement that any Martingale for which the probability of winning is p_0 will yield the return R_0.

Select n gambles such that

$$1-\prod_{i=1}^{n}(1-p_i)=p_0$$

and solve the system of equations

$$\alpha_1(R_1-1)=k\,,\quad \alpha_1(R_1-1)-\sum_{j=1}^{i-1}\alpha_j=k\,,$$

$\Sigma\alpha_i=1$ for the α_i's. Now $\alpha_1=k/R_1-1$, $\alpha_2=k(1/R_1-1+1/(R_s-1)(R_1-1))$ and in general

$$\alpha_m=\frac{1}{R_m-1}+\frac{1}{(R_m-1)(R_{m-1})}+\cdots+$$
$$+\frac{1}{(R_m-1)(R_{m-1}-1)\cdots(R_1-1)}.$$

Thus $\displaystyle\sum_{i=1}^{n}\alpha_i=\left[\prod_{i=1}^{n}\left((1+\frac{1}{R_i-1}\right)-1\right]$

so $\displaystyle k=\frac{1}{\prod_{i=1}^{n}\left(1+\frac{1}{R_i-1}\right)-1}$.

Now from the Martingale relationship

$$R_i=[1-(1-p)^c]^{-1}=\frac{1}{1-(1-p)^c}$$

thus $R_i-R_i(1-p_i)^c=1$, so $\dfrac{R_i-1}{R_i}=$ $=(1-p_i)^c$ and $(1-p_i)^{-c}=\dfrac{R_i}{R_i-1}=1+$ $+\dfrac{1}{R_i-1}$. Therefore

$$k=\frac{1}{\left[\prod_{i=1}^{n}(1-p_i)^{-c}-1\right]}=\frac{1}{(1-p_0)^{-c}-1}=$$
$$=\frac{(1-p_0)^c}{[1-(1-p_0)^c]}=[1-(1-p_0)^c]^{-1}-1\,.$$

So if we forget about the scale factor «-1,» we see that for any Martingale such that $\prod_{i=1}^{n}(1-p_i)=(1-p_0)$, $R(p_0)=[1-(1-p_0)^c]^{-1}$.

The «combination» bet gives an upper limit of $R\leqq p^*R^*/p$ for $p<p^*$, and $R\geqq p^*R^*/p$ for $p>p^*$ but parlay and Martingale limits are always better, so gamblers would never use a combination. To complete the proof we would need to show that given (p^*,R^*),

$$[1-(1-p)^c]^{-1}\leqq p^r \text{ if } p\geqq p^*$$
$$p^r\leqq[1-(1-p)^c]^{-1} \text{ if } p\leqq p^*$$

where

$$r=\frac{\ln R^*}{\ln p^*}\quad c=\frac{\ln(1-R^{*-1})}{\ln(1-p^*)}\,,$$

so that our equilibrium equations make sense. The proof is not relevant to the paper and can be found in [3].

If there were a strong performance for parlay type bets in a market, then the observed relationship between probabilities and returns should be $R=p^r$. Similarly, if the preference where for Martingales, then the relationship would be approximately $R=[1-(1-p)^c]^{-1}$. Since both of these functions are monotone decreasing over (0,1) and thus satisfy the rationality hypothesis, it is impossible to distinguish between these functions as models for actual betting situations on strictly theoretical grounds. Accordingly Rosett took the odds and subsequent wins of 110,000 horses and arranged them (in descending order by odds) into groups of about 350. He then estimated R and p for each group by simple averaging. (If a group did not contain a win, he augmented it until it had one). The resultant 257 pairs of points (p, R) were then fitted to two equations:

$$(a)\quad \ln(1 - R^{-1}) = c_0 + c_1 \ln(1-p) + c_2 \ln^2(1-p),$$

$$(b)\quad \ln R = r_0 + r_1 \ln p + r_2 \ln^2 p .$$

Note that (a) would fit the Martingale relationship exactly if $c_0 = c_2 = 0$ and $c_1 = 1$. Similarly, (b) would fit the parlay relationship if $r_0 = r_2 = 0$ and $r_1 = 1$. Using standard regression analysis, both (a) and (b) were fitted to the data yielding the equations:

$$(a)\quad \ln(1-R^{-1}) = \underset{(0.0216)}{-0.0078} + \underset{(0.02)}{1.15} \ln(1-p) + \underset{(0.03)}{0.09} \ln^2(1-p),$$

(standard deviation)

$$(b)\quad \ln R = \underset{(0.06)}{-0.365} \underset{(0.04)}{-1.27} \ln p \underset{0.007)}{-0.074} \ln^2 p$$

(standard deviation)

Note that c_2 is significantly different from zero, which suggests that the Martingale limit does not hold for all p. Further r_0, r_1, and r_2 are all significantly different from zero suggesting that the parlay relation does not hold for all p.

The data was then re-examined, and it was found that for $p < .02$, the returns were very much lower than would be expected under either model. Removal of these 40 points resulted in the following two equations:

$$\ln R = \underset{(0.08)}{-0.05} - \underset{(0.07)}{0.96} \ln p - \underset{(0.014)}{0.007} \ln^2 p$$

(standard deviation)

$$\ln(1 - R^{-1}) = \underset{(0.24)}{0.0073} + \underset{(0.03)}{1.15} \ln(1-p) + \underset{(0.037)}{0.092} \ln^2(1-p)$$

(standard deviation)

Note that in the first equation only r_1 is significantly different from zero, as are both c_1 and c_2 in the second. The conclusions is that the Martingale relationship does not hold for this set of data, and that the parlay relationship estimated at $R = p^{-0.96}$ holds for $.02 \leq p \leq 1$. Rosett theorizes that the marked deviation from this formula for $p < .02$ is due to either or both of the two phenomena. The first is that some betters bet in a purely random fashion (e. g., bet on numbers, or on names), accordingly placing too much money on very low probability horses, thereby decreasing their return. The second phenomenon is that there exist betters who are so interested in getting a very large return that they always choose the long shots to form parlays. Since this choice is made without consideration of the true probabilities of winning, it has the same effect on low probability bets as does random betting.

The conclusion that Rosett reaches is that gamblers are either relatively sophisticated probabilistically or follow sophisticated advice. I feel that this too strong a statement. Let us note that the empirical relationship is approximately $R = 1/p$ and since the return is really the odds plus one, we see that the betters have been betting in a fashion which might be considered optimum on a purely theoretical basis. A better conclusion to draw for the data then is that betters as a *group* seek to be quite accurate in ascertaining the underlying probability structure of the various races bet upon, but that individually, the betters are not necessarily acting either rationally or in a probabilistically sophisticated fashion.

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MOVIMENTO MATEMÁTICO

CENTRO DE ESTUDOS MATEMÁTICOS DO PORTO

A convite do Instituto de Alta Cultura deslocou-se ao Porto em Março de 1967 o Professor Jean Cerf, especialista de topologia diferencial da Universidade de Paris. Durante a sua estadia de cerca de um mês, este professor realizou um curso sobre o teorema do h-cobordismo (de Smale), assunto que se relaciona com o seu presente trabalho de investigação. Antes da vinda do Professor Jean Cerf, organizou-se no C. E. M. P. uma série de exposições preparatórias que versaram os assuntos seguintes: variedades diferenciáveis (definições gerais), topologia C^r, teorema de Sard (sobre a medida do conjunto dos valores críticos de nma aplicação diferenciável) e o teorema de Morse (sobre a densidade do conjunto das funções cujos pontos críticos são todos não degenerados).

NOTICIÁRIO BRASILEIRO DE MATEMÁTICA

Publicações do IMPA — No decurso de 1968 o IMPA publicou os seguintes trabalhos, no âmbito da Colecção Notas de Matemática:

Malgrange Theorem for Nuclearly Entire Functions of Bounded Type on a Banach Space, by C. Gupta. Notas de Matemática n.º 37.

Suports of Convolutions, by A. Diego. Notas de Matemática n.º 38.

A Theory of interpolation of Normed Spaces, by J. Peetre. Notas de Matemática n.º 39.

Introdução à Teoria das Probabilidades para Matemáticos, por G. Rabson.

Nova morada do IMPA — É a seguinte a nova morada do Instituto de Matemática Pura e Aplicada: Rua Luiz de Camões, 68, Rio de Janeiro 58, GB, Brasil.

J. M. H.

LIÇÕES DE MATEMÁTICA PARA PÓS-GRADUADOS NA FACULDADE DE CIÊNCIAS DE LISBOA

No âmbito do Plano Intercalar de Fomento, efectuaram-se na Faculdade de Ciências de Lisboa duas séries de conferências sobre temas de «Matemática e suas aplicações», com o fim de abrir novos horizontes sobre sectores diversos da problemática actual. As primeiras conferências versarão os seguintes temas:

Matemática Pura: Axiomática dos conjuntos, — pelos Doutor J. Santos Guerreiro e Dr. J. Silva Oliveira; Intuicionismo, pelo Dr. A. Vaz Ferreira;

Matemática Aplicada: Axiomática da Termodinâmica, pelo Prof. Dr. J. Pinto Peixoto; Turbulência e Geometria dos vértices em oceanografia, pelo Ten. Daniel Rodrigues; Computadores e linguagens de programação, pelo Dr. A. Cadete; Optimização pelo Dr. Gustavo de Castro.

J. T. O.