Entropic groupoids and abelian groups

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1. Following ETHERINGTON [1], a groupoid G is said to be *entropic*, if the following condition holds:

$ab \cdot cd = ac \cdot bd$ for all a, b, c, d in G.

ORRIN FRINK [2] observed that every entropic groupoid G has additive endomorphisms, that is to say, if α and β are endomorphisms of G, then the mapping $\alpha + \beta : G \to G$, defined by

$(\alpha + \beta)(x) = \alpha(x) \cdot \beta(x)$ for every $x \in G$,

is an endomorphism of G.

In [1], ETHERINGTON remarked that the converse is not true and gave some examples to show that a groupoid can have additive endomorphisms without satisfying the entropic law.

In [3], TREVOR EVANS proved that if, instead of endomorphisms of G, one considers homomorphisms of the direct product $G \otimes G$ into G, then the property that the sum of two homomorphisms of $G \otimes G$ into G is again a homomorphism, is equivalent to the entropic law.

As an immediate consequence, he stated that, if a loop G has that property, then G is an abelian group.

Thus, it was stated that, *if* G *is an entropic groupoid satisfying the conditions*

(I) *for all* a, b *in* G, *the equations* $ax = b$ *and* $ya = b$ *have unique solutions in* G,

(II) *there exists in* G *an element* e *such that* $ae = ea = a$ *for every* $a \in G$,

then G *is an abelian group.*

The purpose of this note is to show that the conditions (I) and (II) may be weakened.

2. We are going to state the following

THEOREM 1: *Let* G *be an entropic groupoid satisfying the following conditions:*

(i) *For two any elements* $x, y \in G$, *there are elements* $e, f \in G$ *such that*

$$xe = x = fx \quad \textit{and} \quad ye = y = fy;$$

(ii) *For each element* $x \in G$ *and each element* $e \in G$ *such that* $xe = x$, *there is an element* x'_e *(dependent on* e*) such that*

$$x x'_e = e.$$

Then G *is an abelian group.*

PROOF. Indeed, let $x, y \in G$ and let e, f be elements of G satisfying condition (i). Then, by the entropic law, one has

$$xy = fx \cdot ye = fy \cdot xe = yx,$$

that is to say, the groupoid G is commutative.

Moreover, if g denotes any element of G such that

$$xg = x \text{ and } zg = z,$$

then, by the entropic law and the commutativity, one has

$$xy \cdot z = xy \cdot zg = xy \cdot gz = xg \cdot yz = x \cdot yz .$$

Consequently, every entropic groupoid satisfying condition (i) is commutative and associative.

Now, we are going to see that, under the conditions of the theorem, for each element $x \in G$, there is only one local identity, i. e., there is only one element $e \in G$ such that

$$xe = ex = x .$$

In fact, let e and f be elements of G such that

$$xe = xf = x$$

and let x'_e and x'_f be elements of G such that

$$xx'_e = e \text{ and } xx'_f = f .$$

Then, since G is associative and commutative, one has clearly

$$\begin{aligned} e &= xx'_e = xf \cdot x'_e = x \cdot fx'_e = x \cdot x'_e f = \\ &= xx'_e \cdot f = ef = e \cdot xx'_f = ex \cdot x'_f = \\ &= xe \cdot x'_f = xx'_f = f . \end{aligned}$$

It is immediate that e is an identity element of the groupoid G, for, if y is any element of G, from (i) it follows that there is some element f such that $xf = x$ and $yf = y$ and so, by the argument above, one concludes that $f = e$, i. e., one has $ye = ey = y$ for every $y \in G$, meaning that e is an identity element of G.

Now, condition (ii) and commutativity say that every element of G has an inverse element and so G is an abelian group.

Conversely, if G is an abelian group, then G is clearly an entropic groupoid satisfying the conditions (i) and (ii). Thus, an abelian group may be characterized as an entropic groupoid satisfying these conditions.

Remark 1: It is interesting to observe that, if condition (i) is replaced by the condition

(i') *for each element of* G, *there is in* G *one local identity element,*

then G need not be a group.

Indeed, let Q be the set of all rational numbers and let us define in Q the operation $\odot$ by the condition

$$x \odot y = \frac{x+y}{2} \quad \text{for all } x, y \text{ in } Q .$$

It is immediate that the groupoid $\langle Q, \odot \rangle$ is entropic and, moreover, for each $x \in Q$, there is one local identity which is x and there is one local inverse element which is x again.

In spite of that, the groupoid $\langle Q, \odot \rangle$ is not a group.

3. In this section, we obtain another characterization of the abelian groups as entropic groupoids, by using the notion of a center associative element.

Let us recall that an element a of the groupoid G is said to be a *center associative element* (Garrison [4]), if and only if one has

$$x \cdot ay = xa \cdot y \quad \text{for all } x, y \text{ in } G .$$

Theorem 2. *Let* G *be an entropic groupoid satisfying the following condition:*

(C): *There is in* G *some center associative element* a *such that, for every* $b \in G$, *the equations* $ax = b$ *and* $ya = b$ *are soluble in* G. *Then the groupoid* G *is a commutative monoid.*

Proof. One has to prove that G is associative and commutative and, moreover there is in G an identity element.

Let e be a solution of the equation $ax = a$ and let b be any element of G. Since $b = ya$ for some $y \in G$ and a is center associative, one has

$$be = ya \cdot e = y \cdot ae = ya = b,$$

and this means that e is a right identity of the groupoid G.

On the other hand, let e' be a solution of the equation $ya = a$.

Since every $b \in G$ may be written under the form $b = xa$ for some $x \in G$, one has

$$e'b = e' \cdot ax = e'a \cdot x = ax = b$$

and hence e' is a left identity of G.

From $e' = e'e = e$, one concludes that e is an identity element of G.

Now, since G is entropic, it results

$$xy = ex \cdot ye = ey \cdot xe = yx$$

and

$$xy \cdot z = xy \cdot ez = xe \cdot yz = x \cdot yz$$

for all x, y, z in G, which completes the proof.

Remark 2. The hypothesis that G is entropic was not used to show that G has an identity element. Consequently, *every groupoid satisfying condition* (C) *has an identity element.* In particular, *every quasigroup having a center associative element is a loop.*

Theorem 3. *Let* G *be an entropic groupoid satisfying the following condition:*

(D): *There is in* G *some center associative element* a *such that, for every* $b \in G$, *the equations* $ax = b$, $ya = b$ *and* $bz = a$ *are soluble in* G. *Then the groupoid* G *is an abelian group.*

Proof. Since, by theorem 2, G is a commutative monoid, it suffices to show that, for each $b \in G$, there is in G some element b' such that $bb' = e$, where e denotes the identity element.

Let b' be a solution of the equation

$$ab \cdot x = a.$$

Then, one has

$$a = ab \cdot b' = a \cdot bb'.$$

Now, if a' denotes a solution of the equation $ya = e$, one has

$$e = a'a = a'(a \cdot bb') = a'a \cdot bb' = e \cdot bb' = bb',$$

as wanted.

Conversely, if G is an abelian group, then G is obviously an entropic groupoid satisfying condition (D). Thus, by using theorem 3, one obtains another characterization of the abelian groups as entropic groupoids.

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