An absolutely independent axiom system for groups

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1. Introduction

The notion of an *absolutely independent axiom system* was introduced by FRANK HARARI in [1]. Following HARARI, one says that an axiom A of the form «p implies q» *never holds* in a model M, if the hypothesis p occurs at least once in M and, furthermore, whenever p is true, q is false (one admits the possibility of taking p as an universally true statement). Let S be an axiom system and $A \in S$; if there is a model in which. A never holds and every axiom in $S - \{A\}$ holds, then A is said to be *very independent*. The axiom systema S is said to be a *very independent axiom system*, if each of its axioms is very independent. One says that S is an *absolutely independent axiom system*, if for every subset S' of S there is some model in which each axiom belonging to S' never holds and each axiom belonging to $S - S'$ holds.

In general, the usual axiom systems are not absolutely independent.

HARARI gave in [1] an absolutely independent axiom system for equivalence relations and conjectured that some, but not many, others might exist.

J. W. ELLIS presented another independent axiom system in [2], which selects the subgroups among the subsets of a group.

In [3], R. A. JACOBSON and K. L. YOCOM stated that a groupoid $< G, * >$ satisfying the following axioms

(i) *there is an element* e *such that* $x * e = x$ *for all* $x, x \neq e$;

(ii) *for all* $x, y, z, z \neq e$, $x * (y * z) = (x * y) * z$;

(iii) $x * w = y$ *has a unique solution* w *for all* $x, y, x \neq y$,

is a group.

Next, they proved that these axioms constitute an absolutely independent axiom system. (It is supposed that G contains at least three elements).

Neverthless, the axioms (ii) and (iii) are too strong.

The purpose of this note is to weaken the axioms (ii) and (iii) and to obtain another absolutely independent axiom system for groups.

2. A definition of a group

We are going to state the following

THEOREM. *Let* $< G, * >$ *be a groupoid satisfying the conditions:*

A: *There is in* G *some element* e *such that*

$$x * e = x \text{ for every } x \in G - \{e\};$$

B: *If* $x \in G$, $y \in G - \{e\}$ *and* $z \in G - \{e\}$, *then*

$$x * (y * z) = (x * y) * z;$$

C: *There is in* G *some element* a *such that, if* $b \in G - \{a\}$, *then each of the equations*

$$b * x = a \text{ and } a * y = b$$

is soluble in G.

Then the groupoid $<G, *>$ is a group.

PROOF. We have to prove that:

— there is in G some element e such that $x * e = x$ *for every* $x \in G$;

— *for every* $c \in G$ there is an element $c' \in G$ such that $c * c' = e$;

— one has $x * (y * z) = (x * y) * z$ *for all* $x, y, z \in G$.

It is immediate that it suffices to prove the following:

1) *If* $c \in G - \{e\}$, *then there is some* $c' \in G$ *such that* $c * c' = e$.

2) *Oue has* $e * e = e$.

Let us show that the equation $c * z = e$, with $c \neq e$, has at least one solution in G.

If $a = e$, the conclusion follows immediately from the solubility of the equation $b * x = a$ for every $b \neq a$ (condition C).

If $a \neq e$ and $c = a$, the conclusion follows immediately from the solubility of the equation $a * y = b$ for every $b \neq a$ (condition C.

Now, let $a \neq e \neq c \neq a$. Let x and y be elements of G such that $c * x = a$ and $a * y = e$, respectively. One has clearly $x \neq e$ and $y \neq e$, since $x = e$ implies

$$a = c * x = c * e = c \text{ (by condition A)}$$

and $y \neq e$ implies

$$e = a * y = a * e = a \text{ (by condition A)},$$

against the assumptions that $c \neq a$ and $a \neq e$.

Consequently, by condition B, one has

$$e = a * y = (c * x) * y = c * (x * y),$$

that is to say, the element $z = x * y$ is a solution of the equation $c * z = e$, which proves 1).

It is easy to see that

3) *if* $b \neq e$ *and* $b * b' = e$, *then* $b' * b = e$.

In fact, from $b \neq e$ and $b * b' = e$, it follows, by condition A, that $b' \neq e$. Hence, by 1), it results that there is some element b'' in G such that $b' * b'' = e$ and, by condition A, $b'' \neq e$.

If $b' * b \neq e$, then, by condition A,

$$(b' * b) * e = (b' * b) * (b' * b'') = b' * b.$$

Since $b \neq e$, $b' \neq e$ and $b'' \neq e$, it follows, by conditions B and A,

$$b' * b = ((b' * b) * b') * b'' = (b' * (b * b')) * b'' =$$
$$= (b' * e) * b'' = b' * b'' = e.$$

This contradiction proves 3).

Now, we can state the following:

4) *If* $x \in G - \{e\}$, *then* $e * x = x$.

In fact, let x' be an element of G such that $x * x' = e$.

Then, since $x \neq e$ and $x' \neq e$, one has, by condition B and 3),

$$e * x = (x * x') * x = x * (x' * x) = x * e = x,$$

as wanted.

Finally, let us see that $e * e = e$.

Indeed, let $x \in G - \{e\}$ and let x' be an element of G such that $x * x' = e$. Since $x' \neq e$, one has, by condition B and 4),

$$e * e = e * (x * x') = (e * x) * x' = x * x' = e$$

which proves 2) and, consequently the theorem holds.

It is clear that, if the groupoid $<G,*>$ is a group, then the conditions A, B and C hold. Thus, we can say that a group is a groupoid satisfying the conditions A, B, C.

3. Absolute independence of the axiom system S A, B, C

Let us consider the following axioms:

$\bar{A}$: *If* x, y G *and* x y, *then* x y x;

$\bar{B}$: *If* x, y, z, e ∈ G *and* y ≠ e ≠ z, *then*

$$x*(y*z)=(x*y)*z;$$

$\bar{C}$: *If* a, b G *and* a b, *then at least one of the equations*

$$b*x=a \quad \textit{and} \quad a*y=b$$

is not soluble in G.

One sees that the axiom $\bar{A}$ (resp. $\bar{B}, \bar{C}$) holds in $<G,*>$, if and only if the axiom A (resp. B, C) never holds in $<G,*>$.

In order to state the absolute independence of the axiom system S, let us consider the following eight groupoids $<G,*>$ where G is the set of all integers and, in each case, the operation $*$ is defined as it follows:

I) $x*y=x+y$ for all $x, y \in G$;

II) $x*y=y$ for all $x, y \in G$;

III) $x*y=x-y$ for all $x, y \in G$;

IV) $x*y=x$ for all $x, y \in G$;

V) $\begin{cases} x*y=x+1 \text{ for every } x \in G \text{ and } y \in G-\{0\}, \\ x*0=x \text{ for every } x \in G; \end{cases}$

VI) $x*y=|x|+|y|+1$ for all $x, y \in G$;

VII) $\begin{cases} x*y=0 \text{ for every } x \in G-\{0\} \text{ and } y \in G, \\ 0*0=1, \\ 0*y=y \text{ for every } y \in G-\{0\}; \end{cases}$

VIII) $x*y=x+1$ for all $x, y \in G$.

In the first groupoid, the axioms A, B and C hold, i. e., this groupoid is a group; it is easy to see that the axioms $\bar{A}$, B and C hold in the grouproid II), the axioms A, $\bar{B}$ and C hold in the groupoid III), the axioms A, B and $\bar{C}$ in the groupoid IV), the axioms A, $\bar{B}$ and $\bar{C}$ in the groupoid V), the axioms $\bar{A}$, B and $\bar{C}$ in the groupoid VI), the axioms $\bar{A}$, $\bar{B}$ and C in the groupoid VII) and, finally, the axioms $\bar{A}$, $\bar{B}$ and $\bar{C}$ in the groupoid VIII).

The cases I), II), III) and IV) prove that the system S is *very independent.*

The element e, which exists in the cases I), III), IV) and V), is the integer O.

In summary, for each subset S' of S, there is some model in which each axiom belonging to S' never holds and each axiom belonging to $S-S'$ holds, that is to say, S is *an absolutely independent axiom system* for groups.

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