

Note on the definition of a group

by José Morgado

In [1], one deals with the following question:

Is a semigroup⁽¹⁾ G which has a local left identity element (i. e., for each element a in G , there is an element e_a in G , dependent on a , such that $e_a a = a$) and a left inverse element a'_e for each element a in G , such that $a'_e a = e_a$, necessarily a group?

It is shown that such a semigroup need not be a group and it is proved that, if G is a commutative semigroup satisfying the conditions

- (i) for each a in G , there is an element e_a in G such that

$$e_a a = a,$$

- (ii) for each e_a in G such that $e_a a = a$, there is an element a'_e , dependent on e_a , such that

$$a'_e a = e_a,$$

then G is a group.

Next, it is asserted that commutativity is essential to the result.

This is not true.

In fact, if one analyzes the proof given in [1], one sees that the commutativity was not used in full.

In [2], in the section *Advanced Problems and Solutions*, P. J. SALLY proposes the following exercise:

Show that the necessary and sufficient condition for a semigroup G satisfying (i) and (ii) to be a group is that every local left identity is in $C(G)$, the center of G with respect to the semigroup structure.

In [3], it was published a solution for this exercise.

Or, the condition enunciated by P. J. SALLY is too strong.

In fact, one can state the following

THEOREM: Let G be a semigroup satisfying the conditions (i) and (ii). Then G is a group, if and only if the following holds:

- (iii) If e_a and e_b are local left identities relative to a and b , respectively, then

$$e_a e_b = e_b e_a;$$

- (iv) If e_a and i_a are local left identities relative to a , there is some left inverse a'_i of a , relative to i_a , such that

$$e_a a'_i = a'_i e_a.$$

PROOF. Indeed, if G is a group, one has

$$e_a = i_a = e_b$$

and conditions (iii) and (iv) hold.

Conversely, let us suppose that conditions (iii) and (iv) hold.

In order to conclude that G is a group, it suffices clearly to prove that *all the local left identities are equal*.

First, let us observe that for each element $a \in G$, there is only one local left identity.

(1) I. e., an associative groupoid.

