Note on the definition of a group

by José Morgado

In [1], one deals with the following question:

Is a semigroup (1) G which has a local left identity element (i. e., for each element a in G, there is an element e_a in G, dependent on a, such that $e_a a = a$) and a left inverse element a'_e for each element a in G, such that $a'_e a = e_a$, necessarily a group?

It is shown that such a semigroup need not be a group and it is proved that, if G is a *commutative* semigroup satisfying the conditions

(i) *for each* a *in* G, *there is an element* e_a *in* G *such that*

$$e_a a = a,$$

(ii) *for each* e_a *in* G *such that* $e_a a = a$, *there is an element* a'_e, *dependent on* e_a, *such that*

$$a'_e a = e_a,$$

then G is a group.

Next, it is asserted that commutativity is essential to the result.

This is not true.

In fact, if one analyzes the proof given in [1], one sees that the commutativity was not used in full.

In [2], in the section *Advanced Problems and Solutions*, P. J. Sally proposes the following exercise:

Show that the necessary and sufficient condition for a semigroup G *satisfying* (i) *and* (ii) *to be a group is that every local left identity is in* $C(G)$, *the center of* G *with respect to the semigroup structure.*

In [3], it was published a solution for this exercise.

Or, the condition ennunciated by P. J. Sally is too strong.

In fact, one can state the following

Theorem: *Let* G *be a semigroup satisfying the conditions* (i) *and* (ii). *Then* G *is a group, if and only if the following holds:*

(iii) *If* e_a *and* e_b *are local left identities relative to* a *and* b, *respectively, then*

$$e_a e_b = e_b e_a;$$

(iv) *If* e_a *and* i_a *are local left identities relative to* a, *there is some left inverse* a'_i *of* a, *relative to* i_a, *such that*

$$e_a a'_i = a'_i e_a.$$

Proof. Indeed, if G is a group, one has

$$e_a = i_a = e_b$$

and conditions (iii) and (iv) hold.

Conversely, let us suppose that conditions (iii) and (iv) hold.

In order to conclude that G is a group, it suffices clearly to prove that *all the local left identities are equal.*

First, let us observe that *for each element* $a \in G$, *there is only one local left identity.*

(1) I. e., an associative groupoid.

From

$$e_a a_i' = a_i' e_a ,$$

it follows

$$e_a a_i' a = a_i' e_a a ,$$

that is to say,

$$e_a i_a = a_i' a = i_a .$$

Analogously, by (iv), one has

$$i_a a_e' = a_e' i_a ,$$

for some left inverse a_e' of a, relative to e_a; and from this it follows

$$i_a e_a = e_a . \tag{1}$$

Since, by (iii), one has $i_a e_a = e_a i_a$, one concludes that $i_a = e_a$, as it was claimed.

In particular, from (1) it results

$$e_a e_a = e_a . \tag{2}$$

Now, we are going to see that

$$e_a = e_b .$$

In fact, from (2) it follows

$$e_a (e_a e_b) = (e_a e_a) e_b = e_a e_b$$

and

$$e_b (e_b e_a) = (e_b e_b) e_a = e_b e_a ;$$

and, from (iii), it follows

$$e_a (e_a e_b) = e_b (e_a e_b) = e_a e_b .$$

This means that both e_a and e_b are local left identities relative to the element $e_a e_b$, and, therefore, $e_a = e_b$, as it was to be proved.

BIBLIOGRAPHY

[1] W. E. Deskins and J. D. Hill, *On the definition of a group*, Amer. Math. Monthly, **68** (1961), pp. 795-796

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[3] E. A. Schreiner, *Semigroup with Left Inverses and Identities*, Amer. Math. Monthly, **70** (1963), p. 1113.

Sôbre funções de variação total limitada

por *Maria Eulalia Coutinho* *

1. Em [1], pg. 602, (teor. 84), mostra-se que, se f é uma função de variação total limitada definida em um intervalo compacto I de R cujos valores são elementos de um espaço métrico, então existem os limites de f à esquerda e à direita de todo ponto x_0 do interior de I.

Em [2], pg. 110, demonstra-se o seguinte teorema: seja $f: A = [a, b] \times [c, d] \subseteq R^2 \rightarrow R$ uma função de variação total limitada em A tal que, para algum $(x', y') \in A$, as funções $x \rightarrow f(x, y')$ de $[a, b]$ em R e $y \rightarrow f(x', y)$ de $[c, d]$ em R, são de variação total limitada em $[a, b]$ e $[c, d]$ respectivamente. Então, qualquer que seja o ponto (x_0, y_0) do interior de A, existem os quatro limites «angulares» de f quando (x, y) tende para (x_0, y_0). (i. e. existem os limites em cada um dos quadrantes determinados pelas rectas $x = x_0$ e $y = y_0$.

Para esta demonstração é utilizada a de-

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