ANO XXVI — N.º 98-99 GAZETA DE MATEMÁTICA JAN./JUNHO — 1965

EDITOR: *Gazeta de Matemática, Lda.* ADMINISTRADOR: *A. Sá da Costa*

REDACTORES: *J. Gaspar Teixeira, J. Morgado e J. da Silva Paulo*

Composto na Tipografia Matemática, Lda. — Rua Diário de Notícias, 134 - 1.º - Esq. — Telef. 36 94 49 — LISBOA 2

Geodesic curvature of a curve of a vector field

by R. N. Kaul

1. Introduction.

Let V be a vector field in a surface in a Euclidean space of three dimensions. PAN [3] has studied the normal curvature of the vector field V with generalization obtained by the author [2].

The object of the present note is to define the geodesic curvature of the curve of the vector field and obtain some properties. As a special case when the curves of the vector field V form an orthogonal net of co-ordinate curves, these geodesic curvatures have the known form.

2. Consider upon a surface S

$$x^i = x^i(u^1, u^2) \qquad (i = 1, 2, 3),$$

a curve C defined by

$$u^\alpha = u^\alpha(s) \qquad (\alpha = 1, 2).$$

With each point of the surface we associate an arbitrary but fixed vector field V. The components v^i and p^α of the vector field V in the x's and u's are connected by the relation

$$v^i = x^i_{,\alpha}\, p^\alpha .$$

A curve on the surface S along which the vectors of the vector field V are tangent is called the curve of the vector field. It is defined by [3]

$$\varepsilon_{\alpha\beta}\, p^\alpha\, d u^\beta = 0 .$$

The geodesic curvature of the curve C_V of the vector field V shall, therefore, be defined by [1]

$$(2.1)\quad v^k g = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^\beta}\left(\frac{\sqrt{g}\, g^{\alpha\beta}\, \varepsilon_{\lambda\alpha}\, p^\lambda}{(g^{\gamma\delta}\, \varepsilon_{\alpha\gamma}\, \varepsilon_{\mu\delta}\, p^\alpha\, p^\mu)^{1/2}}\right)$$

Use of formulae

$$(2.2)\qquad g^{\alpha\beta}\, \varepsilon_{\alpha\gamma}\, \varepsilon_{\beta\delta} = g_{\gamma\delta}$$

$$(2.3)\qquad \varepsilon_{\lambda\alpha}\, g^{\beta\alpha} = \varepsilon^{\gamma\beta}\, g_{\gamma\lambda}$$

in (2.1) yields the relation

$$v^k g = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^\beta}\left(\frac{e^{\gamma\beta}\, g_{\gamma\lambda}\, p^\lambda}{(g_{\alpha\mu}\, p^\alpha\, p^\mu)^{1/2}}\right).$$

In particular when p^α are the components of a unit vector, we have

$$(2.4)\qquad -v^k g = \varepsilon^{\alpha\beta}\, p_{\alpha;\beta}$$

where semi-colon (;) followed by an index denotes covariant differentiation with respect to u with that index. Since the right hand expression of (2.4) is a scalar called the curl of the vector p_α [3], we have

"The geodesic curvature of the curve C_V of a unit vector field V is a scalar which numerically equals the curl of the covariant components of the vector field V".

Evidently this curl vanishes if p_α are the components of the gradient of a scalar ϕ. Therefore

"The necessary and sufficient condition for the vanishing of the geodesic curvature of a curve of a vector field is that the vector be a gradient".

Suppose now that the vectors of the unit vector field V form an orthogonal net of co-ordinate curves, then from (2.4) we obtain for the geodesic curvatures $v^k g_1$, and $v^k g_2$ of these curves the following relations

$$(2.5)\qquad v^k g_1 = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^2}(g_{11}\,p^1)$$

$$(2.6)\qquad v^k g_2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^1}(g_{22}\,p^2)\,.$$

It is well to recall that

$$p^\alpha = \frac{d\,u^\alpha}{d\,s} \qquad \alpha = 1\,,2$$

and

$$\frac{d\,u^1}{d\,s} = \frac{1}{\sqrt{g_{11}}}\,,\ \frac{d\,u^2}{d\,s} = -\frac{1}{\sqrt{g_{22}}}$$

to verify that (2.5) and (2.6) are the known results for the geodesic curvature of these curves.

Next we consider the orthogonal trajectory C_W of the curve C_V of the vector field V. These are defined by

$$\varepsilon_{\rho\sigma}\, g^{\rho\mu}\, \varepsilon_{\mu\alpha}\, p^\alpha\, d\,u^\sigma = 0$$

which by virtue of (2.2) reduces to

$$g_{\sigma\alpha}\, p^\alpha\, d\,u^\sigma = 0\,.$$

If $w^k g$ denotes the geodesic curvature of the curve C_W, we have

$$w^k g = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^\beta}\left(\frac{\sqrt{g}\, g^{\alpha\beta}\, g_{\lambda\alpha}\, p^\alpha}{(g^{\gamma\delta}\, g_{\alpha\gamma}\, g_{\mu\delta}\, p^\alpha\, p^\mu)^{1/2}}\right)$$

which yields on simplification

$$(2.7)\qquad w^k g = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^\beta}\left(\frac{\sqrt{g}\, p^\beta}{(g_{\alpha\mu}\, p^\alpha\, p^\mu)^{1/2}}\right)$$

Assuming p^α to be a unit vector, we obtain from (2.7)

$$-\,w^k g = \operatorname{div} p^\beta$$

Thus

"The geodesic curvature of the orthogonal trajectory of the curve C_V of a unit vector field V numerically equals the divergence of the vector field".

We can arrive at the above result also by considering a unit vector field $w^i\ (= x^i_{,\alpha}\, q^\alpha)$ orthogonal to the unit vector field $v^i\ (= x^i_{,\alpha}\, p^\alpha)$.

From (2.1), the geodesic curvature of a curve C_W of the vector field W is given by

$$w^k g = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^\beta}(\varepsilon^{\gamma\beta}\, g_{\gamma\lambda}\, q^\lambda)$$

which on using the relations

$$\varepsilon_{\lambda\alpha}\, p^\alpha = q_\lambda$$

$$\varepsilon^{\gamma\beta}\, \varepsilon_{\gamma\mu} = \delta^\beta_\mu$$

yields

$$w^k g = -\operatorname{div} p^\beta\,.$$

Therefore

"The geodesic curvature of a curve of a unit vector field orthogonal to another unit vector field equals numerically the divergence of the latter".

REFERENCES

[1] L. P. Eisenhart, *An introduction to differential geometry with use of the tensor calculus*, Princeton University Press, 1947.

[2] R. N. Kaul, *Generalized normal curvature of a vector field*, The Tensor (New Series). **7**, N.º 3, 1957, pp. 185-189.

[3] T. K. Pan, *Normal curvature of a vector field.* Amer. Jour. Maths., **75**, 1952, pp. 955-966.