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On the maximum value of sums of products*

by J. M. S. Simões Pereira

To Mr. Mário Sousa Santos, in appreciation and affection

1. In this paper we consider finite sets (n_i) of equalsigned real numbers denoted by $n_1, n_2, \cdots$ for which $m \gtrless u$ implies $|n_m| \gtreqqless |n_u|$.

We shall prove the following

THEOREM 1. *Of all sums of* N *products of* j *factors, formed with the* N j *elements of a given set* (n_i) *of real positive numbers taken without repetition, the following one, denoted by* S_+ *will take a maximum value*

$$S_+ = \sum_{p=1}^{p=N} n_{pj-j+1} \cdot n_{pj-j+2} \cdots n_{pj}.$$

In fact, it is always possible to obtain S_+ starting from an arbitrary sum S_1 by interchanging successively the positions of the numbers n_i, these changes being chosen in such a way that the sums intermediately obtained will take nondecreasing values. The aim of such changes will be to assemble the j lowest elements of (n_i) — obtainment of the product $n_1 \cdot n_2 \cdots n_j$ as a term of the sum — then to assemble the j lowest remaining elements of (n_i) in another term — $n_{j+1} \cdot n_{j+2} \cdots n_{2j}$ — and so on.

Fundamentally, one needs to show that one of the sums obtained from

$$S_k = n_1 n_2 \cdots n_k n_u \cdot A + n_{k+1} \cdot B + R$$

— where A and B are products of n_i and R stands for the sum of the remaining terms — by interchanging either n_{k+1} and n_u or $n_1 \cdots n_k$ and k factors of B, will be at least equal to S_k.

Suppose then that we interchange the numbers n_{k+1} and n_u in S_k. We get $S'_{k+1} = n_1 \cdots n_k n_{k+1} A + n_u \cdot B + R$ and it may be $S'_{k+1} \gtreqqless S_k$. Our aim is to get a sum $S_{k+1} \geqslant S_k$, therefore, if $S'_{k+1} < S_k$ we interchange $n_1 \cdots n_k$ and k factors of B, this way we obtain, as we shall see, a sum $S_{k+1} \geqslant S_k$.

In fact, being

$$S'_{k+1} - S_k = (n_{k+1} - n_u)(n_1 \cdots n_k \cdot A - B) < 0,$$

as $n_{k+1} < n_u$, we have $n_1 \cdots n_k \cdot A > B$. Under these conditions and setting $B = N \cdot Q$

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with N standing for a product of k factors, we have

$$S_{k+1}-S_k=(n_1\cdots n_k-N)(n_{k+1}\cdot Q-n_u\cdot A)\geqslant 0.$$

This becomes clear if we remark that $n_1\cdots n_k\leqslant N$ and thus, from $n_1\cdots n_k\cdot A>B=$ $=N\cdot Q$ we get $Q<A$ (positive numbers!); and, on the other hand, multiplying the inequalities $n_{k+1}<n_u$ and $Q<A$ we have $n_{k+1}\cdot Q<n_u\cdot A$.

Plainly, at most $j-1$ operations of this type will suffice to obtain a sum where the term $n_1\,n_2\cdots n_j$ already exists; and then the others $N-1$ terms of S_+ will be successively obtained in a similar way.

S_+ takes the maximum value for we start from an arbitrary sum S_1 and, never decreasing, we get S_+.

The conclusion of this theorem still holds if n_1 is the only n_i negative; in this case when constructing the product $n_1\cdots n_j$ we may always obtain $S_{k+1}\geqslant S_k$ by interchanging n_{k+1} and n_u in S_k.

As a consequence of theorem 1 we can prove:

THEOREM 1 a. *If the elements of* $(\mathrm{n_i})$ *are negative (except perhaps* $\mathrm{n_1}$), *the sum denoted by* $\mathrm{S_+}$ *in the theorem 1 will take a maximum value if the products have an even number of factors and a minimum value if they have an odd one.*

In fact, multiplying all the numbers considered in the THEOREM 1 by -1 we get a set (n_i) in the present conditions; moreover if the number of factors of the products is even all sums will take the same value, if it is odd all will take a simetric one.

Finally, if we are concerned with products of two factors we can prove THEOREM 1 for sets (n_i) of unequalsigned real numbers denoted now in such a way that $m\gtrless u$ implies $n_m\gtreqless n_u$.

The proof is quite imediate; we have found later a similar theorem in [1] but there, it is the case where two sets (n_i) are given that is dealt with.

2. Suppose now that we take a set (n_i) of positive numbers greater than 1 and consider the sums of products of any number of factors that can be formed with these numbers.

Let two sums where the same number of terms will be products of the same number of factors be called of the same type.

We shall prove the following

THEOREM 2. *The maximum sum* $\mathrm{S_+}$ *of all sums of the same type will be obtained by taking the elements* $\mathrm{n_i}$ *in non-decreasing order and forming with them the products in non-decreasing order of the number of factors.*

For example, if there are no products with less than k factors but there are m products with k factors, we must take the $m\cdot k$ lowest n_i and, in accordance with Theorem 1, we shall construct with them the maximum sum of these m products. Then, we consider those of the remaining terms that will have less factors — say, p products of $j\,(>k)$ factors. We take from the remaining n_i the $p\cdot j$ lowest ones and, still in agreement with Theorem 1 we construct the maximum sum of these p products. And so on.

The proof of this theorem is similar to that of Theorem 1. We shall consider two operations that will permit us to obtain S_+ starting from an arbitrary sum S_1 of the same type and we shall show that it is always possible to perform these operations in such a way that the sums intermediately obtained in the process will take nondecreasing values.

Let us investigate these operations.

The aim of one of them is to obtain from a sum

$$S=A\cdot n_z\,n_{z+1}\cdots n_{z+k}\,n_a+B\cdot n_{z+k+1}+R$$

— where we suppose all n_i with $i < z$ already distributed as in S_+ — another sum $S' \geqslant S$ where the product $n_z n_{z+1} \cdots n_{z+k}$ n_{z+k+1} will appear; this being achieved by permutating in S either n_{z+k+1} and n_a or $n_z \cdots n_{z+k}$ and $k+1$ factors of B.

We remark here that with this operation we intend to assemble in the same term, the elements n_z, $n_{z+1}, \cdots n_{z+k}$, n_{z+k+1}; so the terms with less than $k+2$ factors will be already formed like in S_+ and n_{z+k+1} will not appear in anyone of these terms.

For this reason B will be, in fact, a product of at least $k+1$ factors.

Let us interchange then n_{z+k+1} with n_a. We get

$$S_1 = A \cdot n_z \cdots n_{z+k} n_{z+k+1} + B \cdot n_a + R$$

and

$$S_1 - S = (n_{z+k+1} - n_a)(A \cdot n_z \cdots n_{z+k} - B) .$$

As our aim is to obtain a sum $S' \geqslant S$, if $S_1 - S < 0$ we interchange $n_z \cdots n_{z+k}$ with $k+1$ factors of B. In this case we obtain $S_2 = A \cdot N \cdot n_a + B' n_z \cdots n_{z+k} \cdot n_{z+k+1} + R$ where $B = N \cdot B'$ and N is a product of $k+1$ factors, and we can show that $S_2 - S = (n_z \cdots n_{z+k} - N)(B' n_{z+k+1} - A n_a) \geqslant 0$. In fact we have $n_z \cdots n_{z+k} \leqslant N$ and on account of the inequality $A n_z \cdots n_{z+k+1} > B = NB'$ (implied by $S_1 - S < 0$), we get $B' < A$. Multiplying this one by $n_{z+k+1} < n_a$ we obtain $B' n_{z+k+1} < A n_a$ which proves the assertion.

The other operation is concerned with the fact that each term of j factors (for example $n_z n_{z+1} \cdots n_{z+j-1}$) can be constructed within another one which may be a product of more than j factors. As all n_i are greater than 1, if we interchange the product $n_z \cdots n_{z+j-1}$ with another product of j factors as well, say $n_a n_b \cdots n_h$, which is already a term of the initial sum, we get a new sum equal or greater than the former.

In symbols, from

$$S = n_a n_b \cdots n_h + A n_z n_{z+1} \cdots n_{z+j-1} + R$$

where all n_i with $i < z$ are already distributed as in S_+, we get $S' = n_z \cdots n_{z+j-1} + A \cdot n_a \cdots n_h + R$ and $S' - S = (n_z \cdots n_{z+j-1} - n_a \cdots n_h)(1 - A) > 0$ for $n_z \cdots n_{z+j-1} < n_a \cdots n_h$ and $A > 1$.

By means of this two operations we can get the sum S_+ from an arbitrary one, say S_1, through intermediate sums which will take sucessively nondecreasing values and thus Theorem 2 is proved.

REFERENCE

[1] Hardy, Littlewood, Pólya. «*Inequalities*».

On the stochastic convergence of random vectors in real Hilbert space

por *João Tiago Mexia*

1. Introduction

The main objectives of this paper are:

i — to obtain lower bounds of the probability of events that are the intersection of a denumerable or finite family of events, related each one with a random variable.

ii — to study the stochastic convergence of sequences of random vectors as arising from conditions imposed on the sequences of the components with the same index. The case we are mainly interested in is when the vectors have denumerable sets of components although we also consider the case when there is only a finite number of components.