— where we suppose all $n_i$ with $i < z$ already distributed as in $S_+$ — another sum $S' \geqslant S$ where the product $n_z n_{z+1} \cdots n_{z+k}$ $n_{z+k+1}$ will appear; this being achieved by permutating in $S$ either $n_{z+k+1}$ and $n_a$ or $n_z \cdots n_{z+k}$ and $k+1$ factors of $B$.

We remark here that with this operation we intend to assemble in the same term, the elements $n_z$, $n_{z+1}$, $\cdots n_{z+k}$, $n_{z+k+1}$; so the terms with less than $k+2$ factors will be already formed like in $S_+$ and $n_{z+k+1}$ will not appear in anyone of these terms.

For this reason $B$ will be, in fact, a product of at least $k+1$ factors.

Let us interchange then $n_{z+k+1}$ with $n_a$. We get

$$S_1 = A \cdot n_z \cdots n_{z+k} n_{z+k+1} + B \cdot n_a + R$$

and

$$S_1 - S = (n_{z+k+1} - n_a)(A \cdot n_z \cdots n_{z+k} - B).$$

As our aim is to obtain a sum $S' \geqslant S$, if $S_1 - S < 0$ we interchange $n_z \cdots n_{z+k}$ with $k+1$ factors of $B$. In this case we obtain $S_2 = A \cdot N \cdot n_a + B' n_z \cdots n_{z+k} \cdot n_{z+k+1} + R$ where $B = N \cdot B'$ and $N$ is a product of $k+1$ factors, and we can show that $S_2 - S = (n_z \cdots n_{z+k} - N)(B' n_{z+k+1} - A n_a) \geqslant 0$. In fact we have $n_z \cdots n_{z+k} \leqslant N$ and on account of the inequality $A n_z \cdots n_{z+k+1} > B = NB'$ (implied by $S_1 - S < 0$), we get $B' < A$. Multiplying this one by $n_{z+k+1} < n_a$ we obtain $B' n_{z+k+1} < A n_a$ which proves the assertion.

The other operation is concerned with the fact that each term of $j$ factors (for example $n_z n_{z+1} \cdots n_{z+j-1}$) can be constructed within another one which may be a product of more than $j$ factors. As all $n_i$ are greater than 1, if we interchange the product $n_z \cdots n_{z+j-1}$ with another product of $j$ factors as well, say $n_a n_b \cdots n_h$, which is already a term of the initial sum, we get a new sum equal or greater than the former.

In symbols, from

$$S = n_a n_b \cdots n_h + A n_z n_{z+1} \cdots n_{z+j-1} + R$$

where all $n_i$ with $i < z$ are already distributed as in $S_+$, we get $S' = n_z \cdots n_{z+j-1} + A \cdot n_a \cdots n_h + R$ and $S' - S = (n_z \cdots n_{z+j-1} - n_a \cdots n_h)(1 - A) > 0$ for $n_z \cdots n_{z+j-1} < n_a \cdots n_h$ and $A > 1$.

By means of this two operations we can get the sum $S_+$ from an arbitrary one, say $S_1$, through intermediate sums which will take sucessively nondecreasing values and thus Theorem 2 is proved.

### REFERENCE

[1] Hardy, Littlewood, Pólya. «*Inequalities*».

# *On the stochastic convergence of random vectors in real Hilbert space*

por *João Tiago Mexia*

## 1. Introduction

The main objectives of this paper are:

*i* — to obtain lower bounds of the probability of events that are the intersection of a denumerable or finite family of events, related each one with a random variable.

*ii* — to study the stochastic convergence of sequences of random vectors as arising from conditions imposed on the sequences of the components with the same index. The case we are mainly interested in is when the vectors have denumerable sets of components although we also consider the case when there is only a finite number of components.

To accomplish ($i$) we will jointly utilize the technique of passing to the complemental and probability inequalities of the TCHEBYCHEFF type namely the BIENAYMÉ-TCHEBYCHEFF and PEARSON inequalities. To accomplish ($ii$) we take the results pertaining to ($i$) as a point of departure and utilize a technique introduced by TIAGO DE OLIVEIRA [5]. We shall begin with the study of the denumerable analogue of the multinomial case, then we will generalize our results to larger classes of random variables and, using the same technique, obtain new results. Afterwards and through the same methods we will reach specific results concerning the finite case. We end presenting consistent estimators of the quantities introduced.

## 2. A first case

The results presented in this section are contained in the laws of large numbers obtained for randon elements in Banach spaces by EDITH MOURIER [3] but are derived through a much more elementary technique.

Let us start by obtaining some fundamental inequalities; from

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) \leqq \sum_{i=1}^{\infty} \Pr(A_i)$$

and

$$(1) \qquad (A \to B) \to (\Pr(A) \leqq \Pr(B))$$

we have

$$(2) \quad \Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \Pr\left(\bigcup_{i=1}^{\infty} A_i^c\right) \geqq \geqq 1 - \sum_{i=1}^{\infty} \Pr(A_i^c)$$

it is easy to give to (2) the following form with a more general turn:

$$(3) \quad [\forall\, i \to (\Pr(A_i) \geqq 1 - q_i)] \to \to \left(\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) \geqq 1 - \sum_{i=1}^{\infty} q_i\right); (i=1,\cdots,N\cdots).$$

In the finite case:

$$(4) \quad [\forall\, i \to (\Pr(A_i) \geqq 1 - q_i)] \to \to \left(\Pr\left(\bigcap_{i=1}^{N} A_i\right) \geqq 1 - \sum_{i=1}^{N} q_i\right); (i=1,\cdots,N).$$

Let us consider a probability space with a denumerable set of possible outcomes $w_i$ with probabilities «$p_i$», and let «$n_i$» be the number of times that in «$n$» experiences we obtain «$w_i$». We have

$$\Pr\left(\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n} - p_i\right| < \varepsilon\right)\right) \geqq 1 - \sum_{i=1}^{\infty} \Pr\left(\left|\frac{n_i}{n} - p_i\right| \geqq \varepsilon\right) \geqq 1 - \sum_{i=1}^{\infty} \frac{p_i(1-p_i)}{n\,\varepsilon^2} \geqq 1 - \sum_{i=1}^{\infty} \frac{p_i}{n\,\varepsilon^2} = 1 - \frac{1}{n\,\varepsilon^2}$$

as follows from inequalities (2) and from the BIENAYMÉ-TCHEBYCHEFF inequality: so that we obtain:

$$(5) \quad \Pr\left(\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n} - p_i\right| < \varepsilon\right)\right) \geqq 1 - \frac{1}{n\,\varepsilon^2}.$$

Let us study the stochastic limit of:

$$\sum_{i=1}^{\infty}\left|\frac{n_i}{n} - p_i\right|,$$

and prove:

PROPOSITION 1.

$$\sum_{i=1}^{\infty}\left|\frac{n_i}{n} - p_i\right| \xrightarrow{p} 0.$$

P: Let us write:

$$N(\varepsilon) = \underset{N}{\mathrm{Min}} \left\{ \sum_{i=1}^{N} p_i \geqq 1 - \varepsilon/4 \right\};$$

and

$$Y_n = \sum_{i=N(\varepsilon)+1}^{+\infty} \frac{n_i}{n}.$$

It's easy to see that $N(\varepsilon)$ is finite and depends only on «ε» and $Y_n$ has mean value: $\sum_{=N(\varepsilon)+1}^{+\infty} p_i$. We also have

$$\Pr\left(\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n} - p_i\right| < \frac{\varepsilon}{4\,N(\varepsilon)}\right)\right) \geqq$$
$$\geqq 1 - \frac{16\,(N(\varepsilon))^2}{n\,\varepsilon^2};$$

and:

$$\Pr\left(\left|Y_n - \sum_{i=N(\varepsilon)+1}^{+\infty} p_i\right| < \frac{\varepsilon}{4}\right) \geqq 1 - \frac{4}{n\,\varepsilon^2}\ ^{(1)}$$

which are consequences respectively from (5) and from the BERNOULLI theorem. Let us observe that $\sum_{i=1}^{\infty}\left|\frac{n_i}{n} - p_i\right|$ is the sum of the absolute values of positive deviations $\left(\frac{n_i}{n} > p_i\right)$ and of negative deviations: $\left(\frac{n_i}{n} < p_i\right) \cdot \sum_{i=N(\varepsilon)+1}^{+\infty} p_i \leqq \frac{\varepsilon}{4}$, implies that the sum of the negative deviations must be $\leqq \frac{\varepsilon}{4}$; this fact jointly with

$$\left|Y_n - \sum_{i=N(\varepsilon)+1}^{+\infty} p_i\right| < \frac{\varepsilon}{4},$$

implies that the sum of the positive deviations must be $< \varepsilon/2$. As we have

$$\left[\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n} - p_i\right| < \frac{\varepsilon}{4\,N(\varepsilon)}\right)\right] \rightarrow$$
$$\rightarrow \left(\sum_{i=1}^{N(\varepsilon)}\left|\frac{n_i}{n} - p_i\right| < \frac{\varepsilon}{4}\right)$$

we see that

$$\Pr\left\{\left(\left|Y_n - \sum_{i=N(\varepsilon)+1}^{+\infty} p_i\right| < \frac{\varepsilon}{4}\right) \cap\right.$$
$$\left.\cap \left[\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n} - p_i\right| < \frac{\varepsilon}{4\,N(\varepsilon)}\right)\right]\right\} \geqq$$
$$\geqq 1 - \frac{4}{n\,\varepsilon^2}(4\,(N(\varepsilon))^2 + 1).$$

From (4) and the definition of $N(\varepsilon)$ follows

$$\Pr\left\{\left(\left|Y_n - \sum_{i=N(\varepsilon)+1}^{+\infty} p_i\right| < \frac{\varepsilon}{4}\right) \cap\right.$$
$$\left.\cap \left[\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n} - p_i\right| < \frac{\varepsilon}{4\,N(\varepsilon)}\right)\right]\right\} \geqq$$
$$\geqq 1 - \frac{4}{n\,\varepsilon^2}(4\,(N(\varepsilon))^2 + 1)$$

and then, from (1), we get:

$$\text{(6)} \qquad \Pr\left(\sum_{i=1}^{\infty}\left|\frac{n_i}{n} - p_i\right| < \varepsilon\right) \geqq$$
$$\geqq 1 - \frac{4}{n\,\varepsilon^2}(4\,(N(\varepsilon))^2 + 1)$$

which proves the desired result.

---

(1) Let us suppose that the indexes «$i$» where classified, through classifications $C_j$, $j = 1, \cdots, V$, in disjoint subsets $C^*_{j,i}$, and write:

$n^*_{j,i} = \sum_{i \in C_j} n_i$; and $p^*_{j,i} = \sum_{i \in C^*_{j,i}} p_i$. Using first (5) and then (4) we obtain

$$\text{(1)} \quad \Pr\left(\bigcap_{j=1}^{V}\left(\bigcap_{i=1}^{\infty}\left(\left|\frac{n^*_{j,i}}{n} - p^*_{j,i}\right| < \varepsilon_j\right)\right)\right) \geqslant$$
$$\geqslant 1 - \sum_{j=1}^{V} \frac{1}{n\,\varepsilon_j^2}.$$

Let us define now the stochastic convergence of random vectors in an HILBERT space by

(7) $(\vec{a}_n \overset{p}{\to} \vec{a})$ if and only if:

$$\left(\sqrt{\sum_{i=1}^{\infty}(a_{n,i}-a_i)^2} \overset{p}{\to} 0\right)$$

where $\vec{a}_n=(a_{n,i})$, $\vec{a}=(a_i)$. We have

$$\sum_{i=1}^{\infty}|a_i-b_i| \geq \sqrt{\sum_{i=1}^{\infty}(a_i-b_i)^2} \geq 0 \tag{8}$$

whose truth is verified easily by squaring both sides of the inequality. From (7) and (8) follows

$$\left(\sum_{i=1}^{\infty}|a_{n,i}-a_i| \overset{p}{\to} 0\right) \to (\vec{a}_n \overset{p}{\to} \vec{a}) \tag{9}$$

so we have

$$\frac{1}{n}\vec{n} \overset{p}{\to} \vec{p} \tag{10}$$

where $\vec{n}=(n_i)$ and $\vec{p}=(p_i)$. We also have, due to (1), (6) and (5), the following stronger result:

$$\Pr\left(\sqrt{\sum_{i=1}^{\infty}\left(\frac{n_i}{n}-p_i\right)^2}<\varepsilon\right) \geq \tag{11}$$

$$\geq 1-\frac{4}{n\,\varepsilon^2}(4(N(\varepsilon))^2+1).$$

Observing (6) and (11) we conclude the smaller $N(\varepsilon)$ is the better. As $\sum_{i=1}^{\infty} p_i=1$ is a series of non-negative terms we can reorder it. Let $\sum_{j=1}^{\infty}\overline{p}_j=1$, be the series after the terms having been reordered in decreasing order and write

$$\overline{N}(\varepsilon)=\operatorname*{Min}_N\left\{\sum_{j=1}^{N}\overline{p}_j \geq 1-\varepsilon/4\right\};$$

it is easy to see that

$$1 \leq \overline{N}(\varepsilon) \leq N(\varepsilon). \tag{12}$$

As we see that $\forall\, q<1$ we may have $p_1>q$ and whatever $N$ we may have $\overline{p}_1<\frac{1}{N}$ we see that it is impossible to obtain distribution — free bounds for $N(\varepsilon)$ and $\overline{N}(\varepsilon)$.

## 3. Generalization of the results

We will now consider larger classes of a random variables. The PEARSON inequality:

$$\Pr(|x-\mu|<\varepsilon) \geq 1-\frac{\beta_r}{\varepsilon^r} \tag{13}$$

where «$\beta_r$» is the $r$ — th order absolute moment of the deviations of «$x$» from its mean value may be found in SAVAGE [4]. From (3) and (13) we get:

$$\Pr\left(\bigcap_{i=1}^{\infty}(|x_i-\mu_i|<\varepsilon_i)\right) \geq 1-\sum_{i=1}^{\infty}\frac{\beta_{r_i,i}}{\varepsilon^{r_i}} \tag{14}$$

where «$\beta_{r_i,i}$» is the $r_i$ — th order absolute moment of the deviations of «$x_i$» from it's mean «$\mu_i$». Let's now generalize *proposition 1*. We have:

PROPOSITION 2. Let $\{\vec{x}_n\}$, with $\vec{x}_n=(x_{n,i})$, be a sequence of random vectors in a real HILBERT space with non-negative (non-positive) components, whose mean values «$\mu_i$» are independent of «$n$». If, whatever may be «$n$», we have $\sum_{i=1}^{\infty}x_{n,i}=H<+\infty$, and: $\sum_{i=1}^{+\infty} \beta_{r,n,i} \to 0$, where «$\beta_{r,n,i}$» is the $r$-th order absolute moment of the deviations of «$x_{n,i}$» from «$\mu_i$» we have

$$\vec{x}_n \overset{p}{\to} \vec{\mu} \quad \text{where} \quad \vec{\mu} = (\mu_i)$$

P: In KOLMOGOROV [1] it is proven that if the series $\sum_{i=1}^{\infty} |x_i|$ is convergent, the mean value of $\sum_{i=1}^{\infty} x_i$ is $\sum_{i=1}^{\infty} \mu_i^*$, where $\mu_i^*$ is the mean of $x_i$, so that $H = \sum_{i=1}^{\infty} \mu_i$. We can now write

$$N^*(\varepsilon) = \operatorname{Min}_N \left\{ \sum_{i=1}^{N} \mu_i \geqq H - \varepsilon/4 \right\};$$

$$\text{and } y_n = \sum_{i=N^*(\varepsilon)+1}^{+\infty} x_{n,i} = H - \sum_{i=1}^{N^*(\varepsilon)} x_{n,i}.$$

Using

$$\beta_r^3 \geqq \Pr\{|x_{n,i} - \mu_i| \geqq a\} a^r,$$

and

$$\left(\sum_{i=1}^{\infty} \beta_{r,n,i} \to 0\right) \to [\forall\, i \to (x_{n,i} \overset{p}{\to} \mu_i)]$$

we see that $[\forall\, i \to (x_{n,i} \overset{p}{\to} \mu_i)]$. So as $N^*(\varepsilon)$ depends only on «$\varepsilon$»; and as if $g(z_1 \cdots z_N) = u$, where «$g$» is continuous, and $[\forall\, i \to (z_i^n \overset{p}{\to} z_i)]$, then $u^n = g(z_1^n, \cdots, z_N^n) \overset{p}{\to} u$, MEXIA [2]; we have

$$y_n \overset{p}{\to} H - \sum_{i=1}^{N^*(\varepsilon)} \mu_i = \sum_{i=N^*(\varepsilon)+1}^{+\infty} \mu_i$$

so that

$$p_n = \Pr\left\{\left| y_n - \sum_{i=N^*(\varepsilon)+1}^{+\infty} \mu_i \right| \geqq \frac{\varepsilon}{4}\right\} \to 0.$$

Using (14) we obtain

$$\Pr\left(\bigcap_{i=1}^{\infty}\left(|x_{n,i} - \mu_i| < \frac{\varepsilon}{4N^*(\varepsilon)}\right)\right) \geqq$$
$$\geqq 1 - \frac{y^r (N^*(\varepsilon))^r}{\varepsilon^r} \sum_{i=1}^{\infty} \beta_{r,n,i}.$$

Then through the same technique that we used in proving *proposition 1* we obtain:

$$(15) \qquad \Pr\left(\sum_{i=1}^{\infty} |x_{n,i} - \mu_i| < \varepsilon\right) \geqq$$
$$\geqq 1 - \left(\frac{y^r (N^*(\varepsilon))^r}{\varepsilon^r} \sum_{i=1}^{\infty} \beta_{r,n,i} + p_n\right) \underset{n \to +\infty}{\to} 1$$

from (1) and (8) we have:

$$(16) \qquad \Pr\left(\sqrt{\sum_{i=1}^{\infty} (x_{n,i} - \mu_i)^2} < \varepsilon\right) \geqq$$
$$\geqq 1 - \left(\frac{y^r (N^*(\varepsilon))^r}{\varepsilon^r} \sum_{i=1}^{\infty} \beta_{r,n,i} + p_n\right) \underset{n \to +\infty}{\to} 1$$

Our thesis is now proved.

Observing (15) and (16) we see that the smaller $N^*(\varepsilon)$ is the better, as before we can introduce:

$$\overline{N}^*(\varepsilon) = \operatorname{Min}_N \left\{ \sum_{j=1}^{N} \overline{\mu}_j \geqq H - \varepsilon/4 \right\}$$

where the «$\overline{\mu}_j$» are the «$\mu_i$» reordered in

(1) Mantaining the definitions of $C_{j,i}^*$, supposing only that they are finite, and of $p_{j,i}^*$ and writing $y_{j,i}^* = \sum_{i \in C_{j,i}^*} x_i$ we obtain by using first (14) and then (4)

$$(2) \qquad \Pr\left(\bigcap_{j=1}^{V}\left(\bigcap_{i=1}^{\infty}(|y_{j,i}^* - p_{j,i}^*| < \varepsilon_{j,i})\right)\right) \geqslant$$
$$\geqslant 1 - \sum_{j=1}^{V} \sum_{i=1}^{\infty} \frac{\beta_{r_{j,i,j,i}}}{\varepsilon^{r}_{j,i}}$$

where the definition of «$\beta_{r_{j,i,j,i}}$» is analogues to that of $\beta_{r_i,i}$.

decreasing order. We see that:

(17) $$1 \leq \bar{N}^*(\varepsilon) \leq N^*(\varepsilon)$$

We will now obtain a result related to *proposition* 2 but free from the restrictions we imposed on the sign and mean value of the components:

PROPOSITION 3 (1). Let $\{\vec{x}_n\}$, with $\vec{x}_n = (x_{n,i})$, be a sequence of random vectors in an real HILBERT space with means «$\mu_{n,i}$». If there exist positive numbers «$h_{n,i}$» such that for $H_n = \sum_{i=1}^{\infty} h_{n,i}$, and $K_n = \sum_{i=1}^{\infty} \beta_{r,n,i}/h^r_{n,i}$ we have:

$$H^r_n K_n \to 0 \quad n \to +\infty$$

then

$$\sum_{i=1}^{\infty} |x_{n,i} - \mu_{n,i}| \overset{P}{\to} 0$$

and if there are numbers «$a_i$» such that $\sqrt{\sum_{i=1}^{\infty} (\mu_{n,i} - a_i)^2} \to 0$, then $\vec{x}_n \overset{P}{\to} \vec{a}$, where $\vec{a} = (a_i)$.

P: From (14) we have

$$\Pr\left[\bigcap_{i=1}^{\infty} (|x_{n,i} - \mu_i| < \eta_n h_{n,i})\right] \geq$$
$$\geq 1 - \frac{1}{\eta^r_n} \sum_{i=1}^{\infty} \frac{\beta_{r,n,i}}{h^r_{n,i}} = 1 - \frac{K_n}{\eta^r_n}$$

as

$$\left[\bigcap_{i=1}^{\infty} (|x_{n,i} - \mu_i| < \eta_n h_{n,i})\right] \to \left(\sum_{i=1}^{\infty} |x_{n,i} - \mu_{n,i}| < \eta_n H_n\right)$$

using (1) we get

$$\Pr\left(\sum_{i=1}^{\infty} |x_{n,i} - \mu_{n,i}| < \eta_n H_n\right) \geq 1 - \frac{K^n}{\eta^r_n}$$

putting $\eta_n = \varepsilon/H_n$ we obtain

(18) $$\Pr\left(\sum_{i=1}^{\infty} |x_{n,i} - \mu_{n,i}| < \varepsilon\right) \geq$$
$$\geq 1 - \frac{K_n H^r_n}{\varepsilon^r} \to 1 \quad n \to +\infty$$

due to (1) and (8) we then

(19) $$\Pr\left(\sqrt{\sum_{i=1}^{\infty} (x_{n,i} - \mu_{n,i})^2} < \varepsilon\right) \geq$$
$$\geq 1 - \frac{H^r_n K_n}{\varepsilon^r} \to 1 \quad n \to +\infty$$

due to (1) and the triangular property of metric we get:

(20) $$\Pr\left(\sqrt{\sum_{i=1}^{\infty} (x_{n,i} - a_i)^2} < \varepsilon + \sqrt{\sum_{i=1}^{\infty} (a_i - \mu_{n,i})^2}\right) \geq$$
$$\geq 1 - \frac{H^r_n K_n}{\varepsilon} \to 1 \quad n \to +\infty$$

so that we see that out thesis is now proven.

Propositions 2 and 3 can easily be generalized if instead of considering only moments of the $r$-th order we would consider moments of orders $r_{n,i} \leq r$.

We now will prove:

PROPOSITION 4 (1). Let $A$ be an real HILBERT space; «$g$» a continuous function defi-

(1) If we admit that «$g$» is i. e. then it can be shown in the same way that:

(3) $$(\vec{x}_n \overset{P}{\to} \vec{x}) \to (g(\vec{x}_n) \overset{P}{\to} g(\vec{x}))\,.$$

ned in $A$, and $\{\vec{x}_n\}$, where $\vec{x}_n = (x_{n,i})$, a sequence of random vectors of $A$ and such that

*a*) There exists $\vec{a} \varepsilon A$ such that

$$d(\vec{\mu}_n^*, \vec{a}) \to 0,$$

where

$$\vec{\mu}_n^* = (\mu_{n,i}^*),$$

has as components the mean values of the components of $\vec{x}_n$.

*b*) There exist positive numbers «$h_{n,i}$» for which $H_n^r K_n \underset{n\to+\infty}{\to} 0$. Then $g(\vec{x}_n) \overset{p}{\to} g(a)$

D: Due to the fact that «$A$» is a real HILBERT space and to ($a$) and ($b$) we have: $x_n \overset{p}{\to} a$, «$g$» being continuous, for each $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that:

$$(d(\vec{b}, \vec{a}) < \eta(\varepsilon)) \to (|g(\vec{b}) - g(\vec{a})| < \varepsilon)$$

so, due to (1) we have

$$Pr(d(\vec{b}, \vec{a}) < \mu(\varepsilon)) \leq Pr(|g(\vec{b}) - g(\vec{a})| < \varepsilon)$$

so, as $x_n \overset{p}{\to} a$, we see that our result is now proved.

We are now going to show by an example that there are sequences for which the conditions of *proposition 3* are satisfied. Let us take

$$F_{n,i}(x) = \frac{1}{2} H\left(x + \sqrt{\frac{s_i}{u}} - a_i\right) + \frac{1}{2} H\left(x - \sqrt{\frac{s_i}{u}} - a_i\right)$$

with $S = \sum_{i=1}^{+\infty} \sqrt[2r^2]{s_i} < +\infty$ and $H(x)$ such that

$$x < 0 \to H(x) = 0;\ x > 0 \to H(x) = 1$$

we get: $\beta_{r,n,i} = \sqrt[r]{\frac{s_i}{n}}$; and if we write $h_{n,i} = \sqrt[2r^2]{\frac{s_i}{n}}$ we get

$$k_n = \sum_{i=1}^{\infty} \frac{\beta_{r,n,i}}{h_{n,i}^r} = \sum_{i=1}^{\infty} \sqrt[2r]{\frac{s_i}{n}} \leq \leq \sqrt[2r]{\frac{1}{n}}\, S^r \underset{n\to+\infty}{\to} 0$$

and

$$H_n = \sum_{i=1}^{\infty} h_{n,i}^r = \sqrt[2r^2]{\frac{1}{n}}\, S \underset{n\to+\infty}{\to} 0$$

so we have

$$H_n^r K_n \underset{n\to+\infty}{\to} 0.$$

It's clear that the other condition is satisfied.

## 4. The finite case

We are now going to consider results for the finite case. Let «$x_1, \cdots, x_N$» be «$N$» random variables with mean values «$\mu_i$». From (4) and (13) we get

$$Pr\left(\bigcap_{i=1}^{N} (x_i - \mu_i| < \varepsilon_i)\right) \geq 1 - \sum_{i=1}^{N} \frac{Pr_{i,i}}{\varepsilon^{r_i}}$$

and as

$$\left[\bigcap_{i=1}^{N} (|x_i - \mu_i| < \varepsilon_i)\right] \to \left(\sum_{i=1}^{N} |x_i - \mu_i| < \sum_{i=1}^{N} \varepsilon_i\right),$$

due to (1), we get

$$(21)\quad Pr\left(\sum_{i=1}^{N}|x_i-\mu_i|<\sum_{i=1}^{N}\varepsilon_i\right)\geqq 1-\sum_{i=1}^{N}\frac{\beta_{r_{i,i}}}{\varepsilon^{r_i}}.$$

Let's introduce in $R^N$ the euclidean metric

$$d(\vec{a},\vec{b})=\sqrt{\sum_{i=1}^{N}(a_i-b_i)^2}\ .$$

We have

$$\left[\bigcap_{i=1}^{N}\left(|x_{n,i}-a_i|<\frac{\varepsilon}{\sqrt{N}}\right)\right]\rightarrow\left(\sqrt{\sum_{i=1}^{N}(x_{n,i}-a_i)^2}<\varepsilon\right)$$

it's easy to see, using (4), that

$$(22)\quad\left[\bigcap_{i=1}^{N}(x_{n,i}\overset{p}{\rightarrow}a_i)\right]\rightarrow(\vec{x}_n\overset{p}{\rightarrow}\vec{a})$$

where $\vec{x}_n=(x_{n,i})$ and $\vec{a}=(a_i)$.

## 5. Estimation of $N(\varepsilon)$, $\overline{N}(\varepsilon)$, $N^*(\varepsilon)$ and $\overline{N}^*(\varepsilon)$

Let us begin by the estimation of $N(\varepsilon)$. We reorder the fractions $\frac{n_i}{n}$ in decreasing order: $z_{n,i},\cdots,z_{n,j},\cdots$, and write

$$N^n(\varepsilon)=\operatorname*{Min}_N\left[\sum_{j=1}^{N}z_{n,j}\geqq 1-\varepsilon/4-\frac{1}{\sqrt[3]{n}}\right]$$

and

$$\lambda=\frac{1}{2}\left(1-\varepsilon/4-\sum_{j=1}^{N(\varepsilon)-1}\overline{p}_j\right).$$

There exists $\overline{\overline{N}}$ such that

$$n>\overline{\overline{N}}\rightarrow\frac{1}{\sqrt[3]{n}}<L.$$

For $n>\overline{\overline{N}}$ we have

$$\left\{\begin{array}{l}\left[\bigcap_{i=1}^{\infty}\left(\left|\frac{n_i}{n}-p_i\right|<\frac{1}{\sqrt[3]{n}\,\overline{N}(\varepsilon)}\right)\right]\rightarrow\\ \rightarrow\left(\sum_{j=1}^{\overline{N}(\varepsilon)-1}z_{n,j}<1-\frac{\varepsilon}{4}-\frac{1}{\sqrt[3]{n}}<\sum_{j=1}^{\overline{N}(\varepsilon)}z_{n,j}\right)\\ \left(\sum_{j=1}^{\overline{N}(\varepsilon)-1}z_{n,j}<1-\varepsilon/4-\frac{1}{\sqrt[3]{n}}<\sum_{j=1}^{\overline{N}(\ )}z_{n,j}\right)\rightarrow\\ \rightarrow(\overline{N}(\varepsilon)=\hat{\overline{N}}^n(\varepsilon))\end{array}\right.$$

so on account of (1), (5) and the transitive property of implication we get

$$(23)\quad P_r(\hat{\overline{N}}^n(\varepsilon)=\overline{N}(\varepsilon))\geqq 1-\frac{(\overline{N}(\varepsilon))^2}{\sqrt[3]{n}}\underset{n\rightarrow+\infty}{\rightarrow}1$$

If «$s(n)$» is a finite function we have:

$$(24)\quad \underset{n\rightarrow+\infty}{p\lim}(\hat{\overline{N}}^n(\varepsilon)-N(\varepsilon))\,s(n)=0$$

so we see that the asymptotic distribution is a Heaveside distribution.

Using the same technique we obtain:

$$\hat{N}^n(\varepsilon)=\operatorname*{Min}_N\left\{\sum_{i=1}^{N}\frac{n_i}{n}\geqq 1-\varepsilon/4-\frac{1}{\sqrt[3]{n}}\right\}$$

$$\hat{N}^{*n}(\varepsilon)=\operatorname*{Min}_N\left\{\sum_{i=1}^{N}x_{n,i}\geqq H-\varepsilon/4-\sqrt[r']{K_n}\right\}$$

$$\hat{\overline{N}}^{*n}(\varepsilon)=\operatorname*{Min}_N\left\{\sum_{j=1}^{N}Y_{n,j}\geqq H-\varepsilon/4-\sqrt[r']{K_n}\right\}$$

where $r'>2r$ and the «$Y_{n,j}$» are the «$x_{n,i}$» reordered in decreasing order. If $s(n)$ is a finite function we obtain:

$$(25)\quad \begin{aligned}0&=p\lim(\hat{N}^n(\varepsilon)-N(\varepsilon))\,s(n)=\\ &=p\lim(\hat{N}^{*n}(\varepsilon)-N^*(\varepsilon))\,s(n)=\\ &=p\lim(\hat{\overline{N}}^{*n}(\varepsilon)-\overline{N}^*(\varepsilon))\,s(n)\end{aligned}$$

so that the asymptotic distributions are also the Heavside ones.

## 6. References

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# Sobre as várias maneiras de escrever as equações gerais da mecânica dos sistemas com um determinado número finito de graus de liberdade

por *P. de Varennes e Mendonça*

**1. Objectivo** — Ao publicar este artigo só num aspecto o nosso intuito terá acaso excedido objectivos meramente didácticos — o de chamar a atenção para a superioridade formal das equações de Mira Fernandes (*) e de assim procurar fazê-las sair do esquecimento em que injustamente as mantém ainda a maioria dos programas universitários.

**2. Preliminar** — Consideremos o sistema material $C$ sujeito apenas a ligações bilaterais.

Suponhamos ser possível encontrar um número $u$ finito de parâmetros (coordenadas gerais) $q_s (s=1,2,\cdots,u)$ tais que todo o ponto $P \in C$ é função sòmente dos $q_s$ e do tempo $t$, unívoca e bidiferenciável:

$$(1) \qquad P = P(q_1, q_2, \cdots, q_u, t).$$

Então, o deslocamento virtual $\delta P$ de P no instante $t$ tem a expressão

$$(2) \qquad \delta P = \sum_s \frac{\partial P}{\partial q_s} \delta q_s \qquad (s=1,2,\cdots,u)$$

(*) Fernandes, A. de Mira (1940) — *Equazioni della Dinamica*. «Portae Math.» 2: 1-6, 1941.

e o seu deslocamento real $dP$ no intervalo de tempo elementar $dt$ sucessivo ao instante $t$ vale

$$(3) \qquad dP = \sum_s \frac{\partial P}{\partial q_s} d q_s + \frac{\partial P}{\partial t} d t .$$

Sejam as seguintes as $h < u$ equações de ligação (compatíveis e independentes) não consideradas quando da escolha dos $u$ parâmetros $q_s$ (diferenciadas quando holónomas):

$$(4) \qquad \sum_s \varphi_{rs} d q_s + \eta_r d t = 0 \quad (r=1,2,\cdots,h),$$

onde tanto os $\varphi_{rs}$ como os $\eta_r$ são funções de $t$ e dos $q_s$. Às equações (4) correspondem num deslocamento virtual (compatível) de $C$

$$(5) \qquad \sum_s \varphi_{rs} \delta q_s = 0 .$$

O sistema $C$ tem, por conseguinte, $k = u - h$ graus de liberdade.

Tirem-se de (4) os valores de $h$ dos $dq_s$ — por exemplo, os de $d q_{k+1}, d q_{k+2}, \cdots, d q_u$ — e substituam-se em (3). Então, estas equações convertem-se em