é o número de subconjuntos de C_1 com $p-1$ elementos, isto é, $\binom{n-1}{p-1}$.

c) Obtêm-se fórmulas de recorrência nos índices, fazendo no parágrafo 19 $q=n-p$ e as transformações necessárias para obter um índice, à esquerda, igual à soma dos índices, à direita

1) $$\binom{n}{p} = n\ P_{n-1\,|\,p,\,n-p}$$
$$= n \cdot \frac{1}{n-p}\ P_{n-1\,|\,p,\,n-p-1}$$
$$= \frac{n}{n-p}\ P_{n-1\,|\,p,\,n-p-1}$$
$$= \frac{n}{n-p}\binom{n-1}{p}.$$

Com a interpretação fácil: $\binom{n-1}{p}$ é o número de subconjuntos de p elementos em não entra o elemento x; $n\binom{n-1}{p}$ é o número de vezes em que os n elementos não figuram nos subconjuntos de p elementos. Para obter um subconjunto de p elementos é preciso que nele não figurem $n-p$ dos elementos dados. Será então $\frac{n}{n-p}\binom{n-1}{p}$ o número de subconjuntos de p elementos.

2) $$\binom{n}{p} = \frac{1}{p}\ P_{n\,|\,p-1,\,n-p}$$
$$= \frac{1}{p} \cdot n \cdot P_{n-1\,|\,p-1,\,n-p}$$
$$= \frac{n}{p}\ P_{n-1\,|\,p-1,\,n-p}$$
$$= \frac{n}{p} \cdot \binom{n-1}{p-1}$$

Quer dizer: $n \cdot \binom{n-1}{p-1}$ dá o número de vezes que os n elementos figuram nos subconjuntos de p elementos. Para obter um subconjunto de p elementos é preciso utilizar p dessas figurações e será $\frac{n}{p}\binom{n-1}{p-1}$ o número de subconjuntos de p elementos.

(Continua)

Two classroom notes on algebra

by J. J. Dionísio

1. The computation of the Vandermonde determinant.

Let
$$\Delta_n(x_1, \cdots, x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

We have

(1) $$\Delta_2(x_1, x_2) = x_2 - x_1.$$

The computation of $\Delta_n(x_1, \cdots, x_n)$ along the last column by the LAPLACE rule shows that it is a polynomial in x_n of degree $n-1$. Its roots are $x_1, x_2, \cdots, x_{n-1}$. Hence

(2) $$\Delta_n(x_1, \ldots, x_n) = \Delta_{n-1}(x_1, \cdots, x_{n-1}) \prod_{j=1}^{n-1} (x_n - x_j).$$

If we suppose that

$$\Delta_{n-1}(x_1, \cdots, x_{n-1}) = \prod_{i>j} (x_i - x_j)$$

then from (1) and (2) we infer that

$$\Delta_n(x_1, \cdots, x_n) = \prod_{i>j} (x_i - x_j).$$

2. A rule for solving systems of linear equations.

The following result is easily proved.

THEOREM. *Let*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ *matrix with elements in a field* $\mathcal{F}$. *By elementary row operations and interchange of columns it can be reduced to the matrix (over* $\mathcal{F}$*)*

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & b_{1,r+1} & b_{1,r+2} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{2,r+1} & b_{2,r+2} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 & b_{r,r+1} & b_{r,r+2} & \cdots & b_{rn} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $r \leqq min \{m, n\}$ *is the rank of* A.

Now consider the system of linear equations over $\mathcal{F}$

$$(3) \qquad \begin{cases} a_{11}\, x_1 + \cdots + a_{1n}\, x_n = d_1 \\ \qquad \cdots \\ a_{m1}\, x_1 + \cdots + a_{mn}\, x_n = d_m \end{cases}$$

or, in matrix notation,

$$AX = D$$

where

$$A = [a_{ij}], \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad D = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}.$$

Write the matrix

$$A' = [A \mid D],$$

reduce A to the above form B, allowing, however, the elementary row operations act on D as well. (The interchange of columns 1 to r must be accompanied by corresponding interchange of variables). Let the result be the matrix

$$B' = [B \mid D']$$

where

$$D' = \begin{bmatrix} d'_1 \\ \vdots \\ d'_m \end{bmatrix}.$$

The system (3) is solvable if and only if $r = m$ or, if $r < m$,

$$d'_{r+1} = \cdots = d'_m = 0.$$

A solution of the system (3) is then

$$X_0 = \begin{bmatrix} d'_1 \\ \vdots \\ d'_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It is easily seen that the homogeneous linear system associated to (3), that is, in matrix notation, $AX = 0$, has the $v = n - r$ linearly independent solutions

$$X_1 = \begin{bmatrix} -b_{1,r+1} \\ \vdots \\ -b_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -b_{1,r+2} \\ \vdots \\ -b_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots,$$

$$X_v = \begin{bmatrix} -b_{1n} \\ \vdots \\ -b_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$