GAZETA DE MATEMÁTICA
N.º 50 — Dezembro de 1951

On the reversion of series (*)

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One of the most celebrated of Teixeira's discoveries is the extended form of Burmann's theorem which he publisbed (1) in 1900. I propose to show that this theorem may be used to establish a general formula for the reversion of series, or for the calculation of a root of an algebraic equation of any degree.

From the theorem, it is known that if $y(x)$ as a function of x, written in the form

$$y(x) = (x - \alpha)\,\psi(x),$$

then under suitable conditions as regards convergence, we have

$$x - \alpha = \sum_r \frac{\{y(x)\}^r}{r!} \frac{d^{r-1}}{d\alpha^{r-1}} \left[\left\{ \psi(\alpha) \right\}^{-r} \right]$$

Suppose that

$$\begin{aligned} y(x) &= ax + bx^2 + cx^3 + dx^4 + \cdots \qquad (1) \\ &= x(a + bx + cx^2 + dx^3 + \cdots), \end{aligned}$$

so that

$$\alpha = 0$$

$$\psi(x) = a + bx + cx^2 + dx^3 + \cdots. \qquad (2)$$

The theorem becomes

$$x = \sum_{r=1}^{\infty} \frac{\{y(x)\}^r}{r!} \left\{ \frac{d^{r-1}}{dx^{r-1}} \left[\left\{ \psi(x) \right\}^{-r} \right] \right\}_{x=0} \qquad (3)$$

Now it was shown by H. W. Segar (2) that if

$$(a + bx + cx^2 + \cdots)^{-n} = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$$

then the coefficient A_r may by written as a determinant of r rows and columns

$$A_r = \frac{(-1)^r}{r!\,a^{r+n}} \begin{vmatrix} nb & a & 0 & 0 & \cdots \\ 2nc & (n+1)b & 2a & 0 & \cdots \\ 3nd & (2n+1)c & (n+2)b & 3a & \cdots \\ 4ne & (3n+1)d & (2n+2)c & (n+3)b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (4)$$

But from Taylor's theorem we have

$$\left\{ \psi(x) \right\}^{-n} = \sum_r \frac{x^r}{r!} \left\{ \frac{d^r}{dx^r} \left[\psi(x) \right]^{-n} \right\}_{x=0} \qquad (5)$$

Comparing (4) and (5) we have

$$\left\{ \frac{d^r}{dx^r} \left[\psi(x) \right]^{-n} \right\}_{x=0} = \frac{(-1)^r}{a^{r+n}} \cdot \begin{vmatrix} nb & a & 0 & 0 & \cdots \\ 2nc & (n+1)b & 2a & 0 & \cdots \\ 3nd & (2n+1)c & (n+2)b & 3a & \cdots \\ 4ne & (3n+1)d & (2n+2)c & (n+3)b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (6)$$

Substituting from (6) in (3), we have

$$\begin{aligned} x = \frac{y}{a} - b\frac{y^2}{a^3} + \frac{y^3}{3!\,a^5} \begin{vmatrix} 3b & a \\ 6c & 4b \end{vmatrix} - \\ - \frac{y^4}{4!\,a^7} \begin{vmatrix} 4b & a & 0 \\ 8c & 5b & 2a \\ 12d & 9c & 6b \end{vmatrix} + \cdots \\ + \frac{(-1)^{r-1} y^r}{r!\,a^{2r-1}} \begin{vmatrix} rb & a & 0 & 0 & \cdots \\ 2rc & (r+1)b & 2a & 0 & \cdots \\ 3rd & (2r+1)c & (r+2)b & 3a & \cdots \\ 4re & (3r+1)d & (2r+2)c & (r+3)b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \cdots \end{aligned} \qquad (7)$$

This formula (7) is the reversal of the series (1): it gives that root x of the equation (1) which tends to zero when y tends to zero.

(*) Received 1951 February.

(1) *Journal für Math*, CXXII (1900), p. 97.

(2) *Mess. of Math.* XXI (1892), p. 177.