Introduction.

A topological semigroup is a system consisting of a set $S$, an operation $\cdot$, (we omit this dot and write this operation by juxtaposition), and a topology $T$, satisfying the following conditions:

1) for any $x, y \in S$, $xy \in S$;
2) for $x, y, z \in S$, $(xy)z = x(yz)$;
3) the operation $\cdot$ is continuous in the topology $T$.

A topological subsemigroup $H$ of a semigroup $S$ is a topological subspace of $S$ and also a subsemigroup of $S$.

An equivalence relation $R$ defined on a semigroup $S$ is called homomorphic if for any $a, b, c, d \in S$, $aRb$ and $cRd$ imply $acRbd$.

Given an homomorphic equivalence relation $R$ on $S$, we call the set of equivalence classes mod $R$ the quotient set and we denote it by $S/R$.

The mapping from $S$ onto $S/R$ defined by $n(x) = \text{the class mod } R$ to which $x$ belongs is called the natural mapping from $S$ onto $S/R$.

The family $U$ of all subsets $U^*$ of $S/R$ such that $n^{-1}(n(U))$ is also open, where $n$ is the natural mapping from $S$ onto $S/R$.

In this paper we shall prove the following theorems:

**Theorem 1.** If the semigroup $S$ satisfies the condition A if for every open set $U$ of $S$, the subset $n^{-1}(n(U))$ is also open, where $n$ is the natural mapping from $S$ onto $S/R$.

**Theorem 2.** If $S$ and $T$ are two semigroups and $g$ is a homomorphism from $S$ onto $T$, then $g$ induces a homomorphic equivalence relation $R_g$ on $S$.

**Theorem 3.** Let $S$ and $T$ be two topological semigroups and let $g$ be an open homomorphism from $S$ onto $T$. Then

a) $S/R_g$ is a topological semigroup with the quotient topology;
b) the natural mapping \( n \) from \( S \) onto \( S/R_a \) is an open homomorphism;

c) the mapping \( h \) from \( S/R_a \) onto \( T \) defined by \( h(A) = g(a) \) for any \( a \in A \) as a subset of \( S \) and \( A \in S/R_a \) is a topological isomorphism.

**Theorem 4. (The First Isomorphism Theorem).** Let \( S \) and \( T \) be two topological semigroups both satisfying the condition \( A \). Let \( g \) be an open homomorphism from \( S \) onto \( T \) and let \( R^* \) be a homomorphic equivalence relation defined on \( T \). Then there is a homomorphic equivalence relation \( R \) on \( S \) and there is a mapping \( h \) from \( S/R \) onto \( T/R^* \) which is a topological isomorphism.

At the end of the paper, we give an example to illustrate the theorems.

**Theorems**

Theorem 1. If the semigroup \( S \) satisfies the condition \( A \), then the quotient set \( S/R_a \) is a topological semigroup with the quotient topology, and the natural mapping \( n \) from \( S \) onto \( S/R_a \) is an open topological homomorphism.

**Proof.** We have shown that \( S/R_a \) is an abstract semigroup. Now we wish to show that the natural mapping \( n \) from \( S \) to \( S/R_a \) is an abstract homomorphism. Let \( X \) and \( Y \) be two equivalence classes mod \( R_a \), and let \( XY = Z \). Then by definition of the operation in \( S/R_a \), for any \( x \in X \) and \( y \in Y \), \( xy \in Z \). Since the natural mapping \( n \) assigns each element to the class it belongs, we have

\[
n(X) = X, \quad n(Y) = Y, \quad \text{and} \quad n(xy) = n(z) = Z.
\]

These equations together with the equation \( XY = Z \) imply that \( n(xy) = n(x)n(y) \). This shows that the natural mapping \( n \) is an abstract homomorphism from \( S \) onto \( S/R_a \).

Now let \( U^* \) be an open set in \( S/R_a \). By the definition of the quotient topology for \( S/R_a \), \( n^{-1}(U^*) \) is open. Hence \( n \) is continuous.

Let \( U \) be an open set in \( S \). Since \( S \) satisfies the condition \( A \), \( n^{-1}[n(U)] \) is open. Then by the definition of the quotient topology, \( n(U) \) is open.

Now we wish to show that the semigroup operation in \( S/R_a \) is continuous. Let \( A \) and \( B \) be two arbitrary elements in \( S/R_a \) such that \( AB = C \). Suppose that \( W^* \) is an open neighborhood of \( C \). Then \( W = n^{-1}(W^*) \) is an open neighborhood of \( C \), considered as a subset of \( S \). Since the semigroup operation in \( S \) is continuous, for every \( a \in A \) and every \( b \in B \) such that \( ab = c \), there is an open neighborhood \( U_a \) of \( a \) and an open neighborhood \( V_b \) of \( b \) such that \( U_aV_b \subseteq W \).

Choose such a neighborhood \( U_a \) for every \( a \in A \) and such a neighborhood \( V_b \) for every \( b \in B \). Then

\[
\bigcup_{a \in A} U_a V_b = \left[ \bigcup_{a \in A} U_a \right] \left[ \bigcup_{b \in B} V_b \right] \subseteq W.
\]

Now \( \bigcup_{a \in A} U_a \) is an open neighborhood of \( A \) in \( S \), and \( n \) is an open mapping. It follows that \( n \left[ \bigcup_{a \in A} U_a \right] \) is an open neighborhood of the element \( A \) in \( S/R_a \). Similarly \( n \left[ \bigcup_{b \in B} V_b \right] \) is an open neighborhood of the element \( B \) in \( S/R_a \). Since \( \bigcup_{a \in A} U_a \bigcup_{b \in B} V_b \subseteq W \), we have

\[
n \left[ \bigcup_{a \in A} U_a \right] n \left[ \bigcup_{b \in B} V_b \right] = n \left[ \bigcup_{a \in A} U_a \bigcup_{b \in B} V_b \right] \subseteq n(W) = W^*.
\]

Hence we have found an open neighborhood \( n \left[ \bigcup_{a \in A} U_a \right] \) of \( A \) and an open neighborhood \( n \left[ \bigcup_{b \in B} V_b \right] \) of \( B \) such that
This shows that the semigroup operation in $S/R$ is continuous. With this, the proof of the theorem is complete.

**Theorem 2.** If $S$ and $T$ are two semigroups and $g$ is a homomorphism from $S$ onto $T$, then $g$ induces a homomorphic equivalence relation $R_g$ on $S$.

**Proof.** We define a relation $R_g$ on $S$ in the following manner. Suppose that $a$ and $a^*$ are two elements of $S$, then

$$a = a^* \mod R_g$$

if and only if $g(a) = g(a^*)$.

Evidently, $R_g$ is an equivalence relation. We show that $R_g$ is homomorphic, i.e., if $a, a^*, b, b^* \in S$ such that $a = a^* \mod R_g$ and $b = b^* \mod R_g$, then $ab = a^* b^* \mod R_g$.

Now $a = a^* \mod R_g$ implies $g(a) = g(a^*)$, and $b = b^* \mod R_g$ implies $g(b) = g(b^*)$. These two equations imply that $g(a)g(b) = g(a^*)g(b^*)$. Since $g$ is a homomorphism, we have $g(ab) = g(a)g(b)$ and $g(a^*)g(b^*) = g(a^* b^*)$. Hence $g(ab) = g(a^* b^*)$. This means that $ab = a^* b^* \mod R_g$. This completes the proof.

**Theorem 3.** Let $S$ and $T$ be two topological semigroups and let $g$ be an open homomorphism from $S$ onto $T$. Then

a) $S/R_g$ is a topological semigroup with the quotient topology;

b) the natural mapping $n$ from $S$ onto $S/R_g$ is an open homomorphism;

c) the mapping $h$ from $S/R_g$ onto $T$ defined by $h(A) = g(a)$ for any $a \in A$ as a subset of $S$ and $A \in S/R_g$ is a topological isomorphism.

**Proof.** By theorem 2, $g$ induces a homomorphic equivalence relation $R_g$ on $S$. Let $S/R_g$ be the quotient set. Then $S/R_g$ is a semigroup. Let $n$ be the natural mapping from $S$ onto $S/R_g$. We show that the semigroup $S$ satisfies the condition $A$.

Let $U$ be an open subset in $S$. Since $g$ is an open map, $g(U)$ is open in $T$. Also $g$ is continuous. Hence the subset $g^{-1}[g(U)]$ is open in $S$. But $g^{-1}[g(U)] = \{x \in S \mid g(x) = g(y) \text{ for some } y \in U \}$ and $n^{-1}[n(U)] = \{x \in S \mid g(x) = g(y) \text{ for some } y \in U \}$ hence $n^{-1}[n(U)] = g^{-1}[g(U)]$ and $n^{-1}[n(U)]$ is open. This shows that $S$ satisfies the condition $A$.

Since $S$ satisfies the condition $A$, the parts $a)$ and $b)$ follow from theorem 1.

Before proving part c), we wish to show that the mapping $h$ defined in the theorem is well-defined.

Let $A$ be any element of $S/R_g$ and let $a^*$ and $a^{**}$ be any two elements of $A$ as a subset of $S$. Then

$$a^* = a^{**} \mod R_g.$$

This implies

$$g(a^*) = g(a^{**}).$$

Hence

$$h(A) = g(a^*) = g(a^{**}).$$

This shows that $h$ is well-defined.

Also $h$ is a one to one mapping. For each $A \in S/R_g$ there corresponds a unique value

$$h(A) = g(a)$$

in $T$ as shown above. Now since $g$ is a mapping from $S$ onto $T$, for each $t \in T$ there is an element $a \in S$ such that $t = g(a)$, by definition of $R_g$, $a = b \mod R_g$ if and only if $g(a) = g(b)$. It follows that for each $g(a) = t$, there is one and only one equivalence class $A \mod R_g$ such that $h(A) = g(a) = t$. Hence $h$ is a one to one mapping.
We further show that $h$ is an algebraic homomorphism. Let $A$ and $B$ be any two elements in $S/R_g$. Then

$$h(A \cdot B) = g(a \cdot b) = g(a) \cdot g(b) = h(A) \cdot h(B),$$

where $a$ and $b$ are arbitrary elements of $A$ and $B$ respectively. This shows that $h$ is an algebraic homomorphism.

We show also that $h$ is continuous. Let $A$ be an element in $S/R_g$ such that $h(A) = t$, and let $W$ be an open neighborhood of $t$. Since $h(A) = g(a)$ for every $a \in A$, and since $g$ is continuous, for every $a \in A$, there is an open neighborhood $U_a$ of $a$ such that $g(U_a) \subseteq W$. Choose such an open neighborhood $U_a$ for every $a \in A$. Then $\bigcup_{a \in A} (U_a)$ is a neighborhood of $A$ in $S$ and $n\left[\bigcup_{a \in A} (U_a)\right]$ is an open neighborhood of the element $A$ in $S/R_g$. But $g[\bigcup_{a \in A} (U_a)] = h[\bigcup_{a \in A} (U_a)] \subseteq W$. So for any neighborhood $W$ of $h(A)$ of $A$ such that $h\left[\bigcup_{a \in A} (U_a)\right] \subseteq W$. This shows that $h$ is continuous.

Finally we show that $h$ is open. Let $U^*$ be an open subset of $S/R_g$. Since the natural mapping $n$ from $S$ onto $S/R_g$ is continuous, $n^{-1}(U^*)$ is an open subset in $S$. Also, $g$ is an open mapping from $S$ onto $T$. So $g[n^{-1}(U^*)]$ is open in $T$. But

$$g[n^{-1}(U^*)] = h[n^{-1}(U^*)] = h(U^*).$$

Hence $h(U^*)$ is open in $T$. This shows that $h$ is an open mapping. This completes the proof.

We can sum up theorems 1, 2 and 3 by the following form of the fundamental theorem of homomorphism of the topological semigroups:

If the semigroup $S$ satisfies the condition $A$, then the quotient set $S/R$ is a topological semigroup with the quotient topology, and the natural mapping $n$ from $S$ onto $S/R$ is an open topological homomorphism. Conversely, if $g$ is an open homomorphism from $S$ onto a semigroup $T$, then $T$ is topologically isomorphic to the quotient semigroup $S/R_g$, where $R_g$ is a homomorphic equivalence relation defined by $a R_g b$ if and only if $g(a) = g(b); \ a, b \in S$.

**Theorem 4.** (The First Isomorphism Theorem). Let $S$ and $T$ be two topological semigroups both satisfying the condition $A$. Let $g$ be an open homomorphism from $S$ onto $T$ and let $R^*$ be a homomorphic equivalence relation defined on $T$. Then there is a homomorphic equivalence relation $R$ on $S$ and there is a mapping $h$ from $S/R$ onto $T/R^*$ which is a topological isomorphism.

**Proof.** Since $R^*$ is a homomorphic equivalence relation on $T$, by theorem 1, $T/R^*$ is a topological semigroup and the natural mapping $n$ from $T$ onto $T/R^*$ is an open topological homomorphism. Since the mapping $g$ from $S$ onto $T$ is also a homomorphism, it follows that the product mapping $ng$ from $S$ onto $T/R^*$ is also a homomorphism. We show that $ng$ is open. Let $U$ be an open set in $S$. Since $g$ is open, $g(U)$ is open in $T$. Also, $n$ is an open map; so $ng(U)$ is open in $T/R^*$. This shows that $ng$ is an open topological homomorphism.

Now $S$ and $T/R^*$ are two topological semigroups. $S$ satisfies the condition $A$, and $ng$ is an open topological homomorphism from $S$ onto $T/R^*$. Hence, by theorem 2, $ng$ induces a homomorphic equivalence relation $R_{ng}$ and $T/R^*$. Denote $R_{ng}$ by $R$. Then we have $S/R \simeq T/R^*$. We call this isomorphism $h$. This completes the proof.
Example. To illustrate some of the foregoing theorems we give the following example.

Let \((0, \infty)\) be the semigroup of positive real numbers with addition as its operation and with the usual topology as its topology. Let

\[ S = \{(x, y) | x \in (0, \infty), y \in (0, \infty)\} \]

and let the vector addition be defined in \(S\); i.e.,

\[ (x, y) + (x^*, y^*) = (x + x^*, y + y^*) \]

The set \(S\) with the vector addition is a semigroup.

We topologize the semigroup \(S\) with the usual product topology \(P\); i.e., the family

\[ B = \{(U \times V) | U, V \text{ are open in } (0, \infty)\} \]

is the base for the topology \(P\) in \(S\).

We define a relation \(R\) on \(S\) as follows: for \((x, y), (x^*, y^*) \in S\), \((x, y) \sim (x^*, y^*)\) if and only if \(x = x^*\). It is easy to see that this relation \(R\) is an equivalence relation, because the equation \(x = x^*\) is reflexive, symmetric, and transitive. We show that the equivalence relation \(R\) is also homomorphic.

Suppose that \((x_1, y_1), (x_1^*, y_1^*), (x_2, y_2), (x_2^*, y_2^*) \in S\) such that

\[ (x_1, y_1) R (x_2, y_2) \text{ and } (x_1^*, y_1^*) R (x_2^*, y_2^*) \]

Then \(x_1 = x_2\) and \(x_1^* = x_2^*\). From these equations we have

\[ x_1 + x_1^* = x_2 + x_2^* \]

Hence

\[ (x_1 + x_1^*, y_1 + y_1^*) R (x_2 + x_2^*, y_2 + y_2^*) \]

This means that the relation is a homomorphic equivalence relation.

The equivalence classes mod \(R\) are of the form: \(\{x\} \times (0, \infty)\). We denote the set of all equivalence classes mod \(R\) by \(S/R\). We define an operation in \(S/R\) in the following manner. Let \(\{x\} \times (0, \infty)\) and \(\{y\} \times (0, \infty)\) be any two elements in \(S/R\). Then

\[ \{x\} \times (0, \infty) + \{y\} \times (0, \infty) = \{x + y\} \times (0, \infty) \]

Since for any two positive real numbers \(x\) and \(y\) the number \(x + y\) is unique, the operation defined on \(S\) is well-defined. This operation is associative, because the operation of addition in the set of positive real numbers is associative. Hence the set of equivalence classes mod \(R\) with the operation of addition is a semigroup.

We define the natural mapping \(n\) from \(S\) onto \(S/R\) by assigning each element \((x, y)\) to the equivalence class \(\{x\} \times (0, \infty)\). We show that the mapping \(n\) is an algebraic homomorphism. Let \((x, y)\) and \((x^*, y^*)\) be two arbitrary elements in \(S\). Then

\[ n(x, y) = \{x\} \times (0, \infty) \text{ and } n(x^*, y^*) = \{x^*\} \times (0, \infty) \]

and

\[ n((x, y) + (x^*, y^*)) = \{x + x^*\} \times (0, \infty) \]

But

\[ n(x, y) + n(x^*, y^*) = \{x\} \times (0, \infty) + \{x^*\} \times (0, \infty) = \{x + x^*\} \times (0, \infty) \]

Hence

\[ n(x, y) + n(x^*, y^*) = n(x, y) + (x^*, y^*) \]

This shows that the mapping \(n\) is an abstract homomorphism.

Now we topologize the semigroup \(S/R\) with the quotient topology with respect to the mapping \(n\). That is, a subset \(U \times (0, \infty)\) is open in \(S/R\) if and only if \(n^{-1}[U \times (0, \infty)]\) is open in \(S\). We observe that

\[ n^{-1} [U \times (0, \infty)] = U \times (0, \infty) \]

Hence a subset \(U \times (0, \infty)\) of \(S/R\) is open.
if and only if the \( U \) is open in the usual topology of \((0, \infty)\).

If a subset \( U \times V \) is open in \( S \), then the subset

\[
n^{-1}[U \times (0, \infty)] = U \times (0, \infty)
\]

is also open in \( S \). Hence \( S \) satisfies the condition \( A \).

We show that the natural mapping \( n \) from \( S \) onto \( S/R \) is continuous and open. Let \( U \times (0, \infty) \) be an open set in \( S/R \). Then \( n^{-1}[U \times (0, \infty)] \) which equals \( U \times (0, \infty) \) is open in \( S \). Hence \( n \) is continuous. Now let \( U \times V \) be an open subset of \( S \). Then \( n(U \times V) = U \times (0, \infty) \) is open in \( S/R \) according to the observation of the last paragraph. Hence \( n \) is an open mapping.

Finally we show that the semigroup operation in \( S/R \) is continuous. Let \( \{x\} \times (0, \infty) \) and \( \{y\} \times (0, \infty) \) be any two elements in \( S/R \) such that

\[
\{x\} \times (0, \infty) + \{y\} \times (0, \infty) = \{x+y\} \times (0, \infty).
\]

Let \( W \times (0, \infty) \) be an open neighborhood of \( \{x + y\} \times (0, \infty) \). Then since the addition is continuous in the semigroup of positive real numbers, for an open neighborhood \( W \) of \( x + y \), there are open neighborhoods \( U \) of \( x \) and \( V \) of \( y \) such that \( U + V \subseteq W \). Choose \( U \times (0, \infty) \) as an open neighborhood of \( \{x\} \times (0, \infty) \) and \( V \times (0, \infty) \) as an open neighborhood of \( \{y\} \times (0, \infty) \).

Then

\[
U \times (0, \infty) + V \times (0, \infty) = (U + V) \times (0, \infty) \subseteq W \times (0, \infty).
\]

This shows that the semigroup operation in \( S/R \) is continuous.