Cardinality of a set of topologies

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Let $E$ be an infinite set and $K$ be the set of all topologies $\tau$ on $E$, such that $(E, \tau)$ is a Hausdorff compact connected and locally connected space. Question: Which is the cardinal number of the set $K$? The purpose of this note is to answer this question.

1. For any set $Z$, $|Z|$ denotes the cardinal number of $Z$.

Suppose that $K$ is a nonempty set and let $\tau$ be a fixed element of $K$. Since $(E, \tau)$ is a normal connected space, the cardinality of the set $E$ is greater than or equal to $\aleph_0$.

Now, we fix an element $b \in E$. Let us denote by $\tau'$ the product topology on $[0,1] \times E$, where $E$ is topologized with $\tau$ and $[0,1]$ with the usual topology. Put $G = [0,1] \times E - \{(0, b)\}$. $G$ as a subspace of $[0,1] \times E$ is a Hausdorff locally compact connected and locally connected space.

For each partition $|X_1, X_2|$ of $E - \{b\}$ (thus $E - \{b\} = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$), with $|X_1| = |X_2| = |E|$, we fix two bijective functions $f_1: G \rightarrow X_1$ and $f_2: G \rightarrow X_2$. (The set of all partitions of $E - \{b\}$ under the above conditions has cardinality equal to $2^{|E|}$ by virtue of [1] and because $|E \times E| = |E|$.)

Putting $P = |X_1, X_2|$, let us denote by $\tau_P$ the topology on $E$ such that the set $|f_1(Z \cap G) \cup f_2(Z \cap G) \cup \{b\}| \cup \{(0, b) \in Z \in \tau'\} \cup \cup f_1(Z \cap G) \forall Z \in \tau' \cup f_2(Z \cap G) \forall Z \in \tau'$ is an open basis of the topology $\tau_P$.

Proposition 1. $(E, \tau_P)$ is a Hausdorff compact connected and locally connected space.

Proof. It is obvious that $(E, \tau_P)$ is a Hausdorff space. On the other hand, $X_1$ and $X_2$ are open-connected subsets and it follows immediately that $(E, \tau_P)$ is connected. The subspaces $X_1 \cup \{b\}$ and $X_2 \cup \{b\}$ are compact; thus $(E, \tau_P)$ is a compact space.

Finally, it is sufficient to prove that $b$ has a fundamental system of connected neighborhoods. Let $W$ be a connected neighborhood of $b$ in the topological space $(E, \tau)$ and put $Z = [0,1] \times W$, where $n \geq 1$ is a natural number. The set $Z$ is a connected neighborhood of $b$ in the product topological space $([0,1] \times E, \tau')$. Therefore $f_1(Z \cap G) \cup f_2(Z \cap G) \cup \{b\}$ is a connected neighborhood of $b$ in the topological space $(E, \tau_P)$. So it follows easily that $(E, \tau_P)$ is locally connected.

2. Let $Q = |Y_1, Y_2|$ be another partition of $E - \{b\}$, with $|Y_1| = |Y_2| = |E|$, and let $\tau_Q$ be the correspondent topology on $E$.

We shall prove that if $P \neq Q$, then $\tau_P \neq \tau_Q$.

On the contrary, suppose that $P = Q$ and $\tau_P = \tau_Q$. The sets $X_1, X_2, Y_1$ and $Y_2$ are open-connected in $(E, \tau_P)$ and $X_1 \cup X_2 = Y_1 \cup Y_2$. Thus $X_1 = Y_1$ and $X_2 = Y_2$ or $X_1 = Y_2$ and $Y_1 = X_2$. It follows that $P = Q$, which is impossible.

Proposition 2. The cardinality of the set $K$ is equal to 0 or to $2^{|E|}$.

Proof. Since for any $\sigma \in K, (E, \sigma)$ is Hausdorff compact, then $|K|$ is less than or equal to $2^{|E|}$. So the proof is completed.

References