On the condition \((xy)^2 = x^2y^2\) in nonassociative rings

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1. It is well known that a group \(G\) satisfying the condition

\[(xy)^2 = x^2y^2\] for all \(x, y\) in \(G\)

is necessarily commutative.

It is also well known that if \(G\) is a group such that

\[(xy)_i = x^iy^i\] for three consecutive integers \(i\) and for all \(x, y\) in \(G\),

then \(G\) is necessarily commutative ([1], p. 31, exercise 4).

However, the ring-theoretic analogues of these group-theoretic results do not hold.

Thus, McCoy ([2], p. 15, example 6 and p. 31, exercise 7) gives an example of a noncommutative ring satisfying the condition (1) and E. C. Johnsen, D. L. Outcalt and A. Yaquub [3] give an example of a noncommutative ring satisfying the condition (2).

In [3], the authors prove that if \(R\) is any nonassociative (i.e., not necessarily associative) ring with identity satisfying the condition (1), then \(R\) is commutative.

The purpose of this note is to generalize this result.

2. Let us recall that an element \(a\) of a multiplicative groupoid \(G\) is said to be an associative element, if one has

\[ax \cdot y = a \cdot xy,\] \[xa \cdot y = x \cdot ay\]

and

\[xy \cdot a = x \cdot ya\]

for all \(x, y\) in \(G\). The element \(a\) is said to be cancellable, if for all \(x, y\) in \(G\) each of the equations \(ax = ay\) and \(xa = ya\) implies \(x = y\).

We are going to state the following

**Theorem:** Let \(R\) be any nonassociative ring satisfying the condition

\[(xy)^2 = x^2y^2\] for all \(x, y\) in \(R\).

If for every \((x, y) \in R \times R\) there is in \(R\) some element \(e\) which is associative and cancellable in the multiplicative subgroupoid of \(R\) generated by the set \(\{x, y, e\}\), then \(R\) is commutative.

**Proof:** Indeed, one has obviously

\[(xy + z)^2 = (xy + xz)^2 = xy \cdot xy + xy \cdot xz + xz \cdot xy + xz \cdot xz\]

for all \(x, y, z\) in \(R\).
On the other hand, by taking into account the hypothesis of the theorem, one sees that

\[(x(y+z))^2 = x^2(y+z)^2 =
= x^2(y^2 + yz + zy + z^2) =
= x^2 y^2 + x^2 y z + x^2 z y + x^2 z^2.
\]

Consequently, one has

\[(3) \quad x^2 y z + x^2 y z = x y x z + x z x y \]
for all \(x, y, z \in R\).

If one starts with \(((x+z)y)^2\), one obtains, by a similar way,

\[(4) \quad x y z y + z y x y = x z y^2 + z x y^2 \]
for all \(x, y, z \in R\).

By changing \(x\) to \(x+z\) in (3), one gets

\[
x^2 y z + x z y z + x z y z + z y z +
+ x^2 y z + x z y z + x y z^2 +
+ z y x z + z y z y + x y z + x^2 z y
\]
and, hence, it results by (3),

\[(5) \quad x z y z + x z y z + x z y z =
= x y z^2 + z y x z + z y z^2 + z^2 x y \]
for all \(x, y, z \in R\).

Now, we are going to show that the element \(e\) satisfying the conditions required in the theorem, commutes with \(x\) and \(y\).

In fact, by putting \(x=z=e\) in (3), it results

\[e^2 y e + e^2 y e = e y e^2 + e^2 e y\]
and so

\[(6) \quad e^2 y e = e y e^2.\]

Since \(e\) is associative in the multiplicative subgroupoid generated by \(\{x, y, e\}\), the equality (6) implies

\[e^2 y e = (e y e) e\]
and, since \(e\) is cancellable in that groupoid, it follows

\[e^2 y = e y e.\]

By repeating the argument, one obtains

\[e y = y e,\]
as wanted.

Analogously, by putting \(y=z=e\) in (4), one gets \(e x = x e\).

If in (5) one puts \(z=e\), it results by (6),

\[e x y e + e x y e + e x e y =
= x y e^2 + e y x e + e^2 x y e.\]

Or, one has

\[x e y e = (x e y) e = (x y e) e = (x y e) e =
= (x y e) e = x y e^2,\]

by the associativity of \(e\) and commutativity of \(e\) with \(y\).

Similarly, one sees that

\[e x y = e^2 x y\]
and, consequently, from (7) it follows

\[e x y e = e y x e\]
and, hence,

\[e x y e = (e y x) e.\]

Since \(e\) is cancellable, one concludes

\[e x y = e y x,\]

hence,

\[e x y = e y x,\]
that is to say,

\[x y = y x,\]
as it was to be proved.

**BIBLIOGRAPHY**

