A note on the normal endomorphisms of a group

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1. It is well known that in an abelian group, for every integer $n$, the mapping $\overline{n}: x \mapsto x^n$ is an endomorphism.

In [1], E. Schenkman and L. I. Wade have considered the converse question whether a group is abelian when $\overline{n}$ is an endomorphism. One knows that, if there are three consecutive integers $i$ for which the mappings $x \mapsto x^i$ are endomorphisms, then the group is abelian. However, from the fact that the mappings $x \mapsto x^i$ and $x \mapsto x^{i+1}$ are endomorphisms for some integer $i$, one cannot conclude that the group be abelian ([2], Exercises 4 and 5, p. 31).

Let $G$ be a group and let $G \{n\}$ be the subgroup of $G$ generated by all elements whose orders divide $n$. In [1], it is stated that

1) if $\overline{n}$ is an endomorphism, then $G/G \{n^2-n\}$ is abelian;

2) if $\overline{n}$ is an automorphism, then $G/G \{n-1\}$ is abelian;

and, consequently,

3) if $G$ has no elements whose orders divide $n^2-n$ or if $G$ has no elements whose orders divide $n-1$ when $\overline{n}$ is an automorphism, then $G$ is abelian.

The purpose of this note is to improve the results obtained by Schenkman and Wade.

2. Let us recall that an endomorphism $\alpha$ of a group $G$ is said to be a normal endomorphism of $G$, if $\alpha$ commutes with every inner automorphism of $G$, i.e., if one has

$$\alpha(xyx^{-1}) = x \alpha(y) x^{-1}$$

for all $x, y$ in $G$.

Since $(xyx^{-1})^n = x y^n x^{-1}$ for all $x, y$ in $G$, one sees that, if $\overline{n}$ is an endomorphism of $G$, then it is necessarily a normal endomorphism.

The identity operator of $G$ will be denoted by $\varepsilon$ and by $\varepsilon - \alpha$ one means, as it is natural, the operator of $G$ defined by

$$(\varepsilon - \alpha)(x) = x \alpha(x^{-1}).$$

In general, the operator $\varepsilon - \alpha$ is not an endomorphism, as one concludes from the following

**Theorem 1.** Let $\alpha$ be an endomorphism of the group $G$. Then $\varepsilon - \alpha$ is an endomorphism, if and only if $\alpha$ is normal. Moreover, if $\alpha$ is a normal endomorphism, then the endomorphism $\varepsilon - \alpha$ is normal.

**Proof.** Indeed, one has

$$(\varepsilon - \alpha)(xy) = x y \alpha(y^{-1}x^{-1}) = x y \alpha(y^{-1}) \alpha(x^{-1})$$

for all $x, y$ in $G$.

On the other hand,

$$(\varepsilon - \alpha)(x) \cdot (\varepsilon - \alpha)(y) = x \alpha(x^{-1}) \cdot y \alpha(y^{-1}).$$

Consequently, $\varepsilon - \alpha$ is an endomorphism, if and only if

$$y \alpha(y^{-1}) \alpha(x^{-1}) = \alpha(x^{-1}) y \alpha(y^{-1}),$$
that is to say, if and only if
\[ \alpha(y^{-1})\alpha(x^{-1})\alpha(y) = y^{-1}\alpha(x^{-1})y, \]
for all \( x, y \) in \( G \).

This means that \( \varepsilon - \alpha \) is an endomorphism, if and only if one has
\[ \alpha(y^{-1}x^{-1}y) = y^{-1}\alpha(x^{-1})y \]
for all \( x, y \) in \( G \),

which proves the first part of the theorem.

Moreover, one has clearly, for all \( x, y \) in \( G \),
\[ y(\varepsilon - \alpha)(x)y^{-1} = yx\alpha(x^{-1})y^{-1} = yx y^{-1} \alpha(x^{-1})y^{-1} = yx y^{-1}\alpha(y x^{-1}y^{-1}) = (\varepsilon - \alpha)(yx y^{-1}), \]
proving that \( \varepsilon - \alpha \) is normal.

**Theorem 2.** If \( \alpha \) is a normal endomorphism of the group \( G \), then \( \alpha - \alpha^2 \) is a normal endomorphism of \( G \) and the quotient group \( G / \text{Ker}(\alpha - \alpha^2) \) is abelian.

**Proof.** By theorem 1, the operator \( \varepsilon - \alpha \) is a normal endomorphism.

It is immediate that, if \( \alpha \) and \( \beta \) are normal endomorphisms, then the composite endomorphism \( \alpha \circ \beta \) is also normal.

Since
\[ \alpha - \alpha^2 = \alpha \circ (\varepsilon - \alpha), \]
one sees that \( \alpha - \alpha^2 \) is a normal endomorphism.

In order to show that the quotient group \( G / \text{Ker}(\alpha - \alpha^2) \) is an abelian group, it is sufficient to show that all commutators of \( G \) are in the kernel of the endomorphism \( \alpha - \alpha^2 \), that is to say, for all \( x, y \) in \( G \),
\[ (\alpha - \alpha^2)(xy x^{-1}y^{-1}) = e, \]
where \( e \) denotes the neutral element of \( G \).

Or, by the normality of \( \alpha \), one has obviously
\[ \alpha(xy x^{-1}y^{-1}) = \alpha(x)\alpha(y)\alpha(x^{-1})\alpha(y^{-1}) = \alpha(x)\alpha(\alpha(y)x^{-1}\alpha(y^{-1})) = \alpha(x)\alpha^2(y)\alpha(x^{-1})\alpha^2(y^{-1}) = \alpha(x)\alpha^2(y)\alpha^2(x^{-1})\alpha^2(y^{-1}) = \alpha^2(x)\alpha^2(y)\alpha^2(x^{-1})\alpha^2(y^{-1}) = \alpha^2(xy x^{-1}y^{-1}). \]

From this it follows
\[ \alpha(xy x^{-1}y^{-1})\alpha^2((xy x^{-1}y^{-1})^{-1}) = e \]
and, consequently,
\[ (\alpha - \alpha^2)(xy x^{-1}y^{-1}) = e, \]
as it was to be proved.

**Corollary 1.** If \( \alpha \) is an injective normal endomorphism of the group \( G \), then the quotient group \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian.

In fact, from
\[ \alpha(xy x^{-1}y^{-1}) = \alpha^2(xy x^{-1}y^{-1}), \]
it results, since \( \alpha \) is injective,
\[ xy x^{-1}y^{-1} = \alpha(xy x^{-1}y^{-1}) \]
and hence
\[ (\varepsilon - \alpha)(xy x^{-1}y^{-1}) = e \]
for all \( x, y \) in \( G \), proving that \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian.

**Corollary 2.** If the mapping \( \bar{n} : x \mapsto x^n \) is an endomorphism of the group \( G \), then the quotient group \( G / G | \text{n}^2 - n| \) is abelian. If, moreover, the endomorphism \( \bar{n} \) is injective, then \( G / G | n - 1| \) is abelian.

In fact, \( \bar{n} \) is normal and one has clearly
\[ G | n^2 - n| = \text{Ker}(\bar{n} - \bar{n}^2) \]
and, if \( \bar{n} \) is injective, then
\[ G | n - 1| = \text{Ker}(1 - \bar{n}), \]
where \( 1 \) denotes the identity endomorphism.
3. Now, let us suppose that the endomorphism \( \alpha \) is such that
\[ x \alpha(x^{-1}) \in Z \]
for every \( x \in G \),
where \( Z \) denotes the center of \( G \).

It is easy to see that \( \varepsilon - \alpha \) is normal.

Indeed, since \( \alpha(x^{-1}) = x^{-1}z \) for some \( z \in Z \),
one has
\[ \alpha(x y^{-1}) = \alpha(x) \alpha(y) \alpha(x^{-1}) = x^{-1} x \alpha(y) x^{-1} z = x \alpha(y) x^{-1} \]
for all \( x, y \) in \( G \), proving that \( \alpha \) is normal.

Then, by Theorem 1, one concludes that \( \varepsilon - \alpha \) is a normal endomorphism of \( G \).

We are going to see that the quotient group \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian.

In fact, since
\[ (\varepsilon - \alpha)(x y^{-1} y^{-1}) = x y x^{-1} y^{-1} \alpha(y) x y x^{-1} x^{-1} = x y x^{-1} y^{-1} \alpha(y) \alpha(x) \alpha(y^{-1}) \alpha(x^{-1}) = x y x^{-1} x \alpha(x) y^{-1} \alpha(y) \alpha(y^{-1}) \alpha(x^{-1}) = x y x^{-1} x \alpha(x) y^{-1} \alpha(x^{-1}) = x y y^{-1} x^{-1} \alpha(x) \alpha(x^{-1}) = e \]
for all \( x, y \) in \( G \), one sees that the kernel of \( \varepsilon - \alpha \) contains the subgroup generated by the commutators and, therefore, the quotient group \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian.

Conversely, let us suppose that \( \varepsilon - \alpha \) is a normal endomorphism and \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian.

Then, one has
\[ x y x^{-1} y^{-1} \alpha(y) \alpha(x) \alpha(y^{-1}) \alpha(x^{-1}) = e \]
for all \( x, y \) in \( G \).

From this it follows
\[ x y x^{-1} y^{-1} \alpha(y) = \alpha(x) \alpha(y) \alpha(x^{-1}) = \alpha(x y x^{-1}) = x \alpha(y) x^{-1} \]
in view of the fact that \( \alpha \) is a normal endomorphism, by Theorem 1 and \( \alpha = \varepsilon - (\varepsilon - \alpha) \).

Hence
\[ y x^{-1} y^{-1} \alpha(y) = \alpha(y) x^{-1} \]

Consequently,
\[ x^{-1} y^{-1} \alpha(y) = y^{-1} \alpha(y) x^{-1} \]
for all \( x, y \) in \( G \).

This means that \( y^{-1} \alpha(y) \in Z \) for every \( y \in G \).

In short, the following holds:

**Theorem 3.** If \( \alpha \) is an endomorphism of the group \( G \) such that \( x \alpha(x^{-1}) \) is in the center of \( G \) for every \( x \in G \), then \( \varepsilon - \alpha \) is a normal endomorphism and \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian; conversely, if \( \varepsilon - \alpha \) is a normal endomorphism and \( G / \text{Ker}(\varepsilon - \alpha) \) is abelian, then \( \alpha \) is an endomorphism such that \( x \alpha(x^{-1}) \) is in the center of \( G \) for every \( x \in G \).

In particular, one has the following

**Corollary.** If \( \alpha \) is a central endomorphism of the group \( G \), i.e., if \( \alpha(x) \in Z \) for every \( x \in G \), then the quotient group \( G / \text{Ker}(\alpha) \) is abelian.

Indeed, it is immediate that \( \alpha \) is a normal endomorphism and so \( \varepsilon - \alpha \) is also a normal endomorphism.

One has, for every \( x \in G \),
\[ \alpha(x) = x x^{-1} \alpha(x) = x (\varepsilon - \alpha)(x^{-1}) \in Z \]
and the conclusion follows immediately from Theorem 3.

**BIBLIOGRAPHY**

