A theorem on abelian quotient groups of a group

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1. In a previous paper [1], we have shown that, if \( \alpha \) is a normal endomorphism of the group \( G \), then the operator \( \alpha - \alpha^2 \) is also a normal endomorphism of \( G \) and the quotient group \( G/\text{Ker}(\alpha - \alpha^2) \) is an abelian group.

In this note, we extend this result; we state a necessary and sufficient condition in order that the quotient group \( G/\text{Ker}(\beta - \beta \alpha) \) be an abelian group, \( \alpha \) and \( \beta \) being endomorphisms of \( G \) and \( \beta \alpha \) being the composite of \( \beta \) and \( \alpha \).

2. It is well known that, if \( \alpha \) and \( \beta \) are endomorphisms of the group \( G \), then the operator \( \beta - \alpha \) of \( G \), defined by

\[
(\beta - \alpha)(x) = \beta(x)\alpha(x^{-1})
\]

for every \( x \in G \), need not be an endomorphism of \( G \).

In fact, since

\[
(\beta - \alpha)(xy) = \beta(xy)\alpha(y^{-1}x^{-1}) = \beta(x)\beta(y)\alpha(y^{-1})\alpha(x^{-1})
\]

and, on the other hand,

\[
(\beta - \alpha)(x)(\beta - \alpha)(y) = \beta(x)\alpha(x^{-1})\beta(y)\alpha(y^{-1}),
\]

one concludes that \( \beta - \alpha \) is an endomorphism, if and only if one has

\[
\beta(y)\alpha(y^{-1})\alpha(x^{-1}) = \alpha(x^{-1})\beta(y)\alpha(y^{-1})
\]

for all \( x, y \) in \( G \).

This means that the following holds:

**Lemma.** If \( \alpha \) and \( \beta \) are endomorphisms of the group \( G \), then the operator \( \beta - \alpha \) is an endomorphism of \( G \), if and only if the image of \( \beta - \alpha \) is in the centralizer of the image of \( \alpha \) in \( G \).

**Theorem.** Let \( \alpha \) and \( \beta \) be endomorphisms of the group \( G \). Then the operator \( \beta - \beta \alpha \) is an endomorphism of \( G \) and the quotient group \( G/\text{Ker}(\beta - \beta \alpha) \) is abelian, if and only if the image of \( \beta - \beta \alpha \) is contained in the center of the image of \( \beta \).

**Proof.** Let \( \beta - \beta \alpha \) be an endomorphism. Then, as it is well known, the quotient group \( G/\text{Ker}(\beta - \beta \alpha) \) is abelian, if and only if the kernel of the endomorphism \( \beta - \beta \alpha \) contains the commutator subgroup of \( G \), that is to say,

\[
1 \quad (\beta - \beta \alpha)(xyx^{-1}y^{-1}) = e
\]

for all \( x, y \in G \), \( e \) being the neutral element of \( G \).

First, let us suppose that one has

\[
2 \quad \text{Im}(\beta - \beta \alpha) \subseteq \text{Center of Im}(\beta).
\]

Since

\[
\text{Center of Im}(\beta) \subseteq \text{Centralizer of Im}(\beta \alpha) \text{ in } G,
\]

one concludes by Lemma above that the operator \( \beta - \beta \alpha \) is an endomorphism of \( G \).

Furthermore, one has

\[
(\beta - \beta \alpha)(xyx^{-1}y^{-1}) = \\
= \beta(xy)(x^{-1}y^{-1})\beta \alpha(xy)(x^{-1}y^{-1}) = \\
= \beta(x)\beta(y)\beta(x^{-1})\beta(y^{-1})
\]

for all \( x, y \) in \( G \).

This proves (1).

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Now, let us suppose that the operator $\beta - \beta \alpha$ is an endomorphism of the group $G$ such that the quotient group $G/\text{Ker}(\beta - \beta \alpha)$ is abelian.

From (1) it follows

$$(\beta - \beta \alpha)(xy)(\beta - \beta \alpha)(x^{-1}y^{-1}) = e,$$

hence

$$(\beta - \beta \alpha)(x)(\beta - \beta \alpha)(y) = (\beta - \beta \alpha)(yx),$$

that is to say,

$$\beta(x)\beta(x^{-1})\beta(y)\beta(y^{-1}) = \beta(y)\beta(x)\beta(x^{-1})\beta(y^{-1}).$$

Consequently,

$$\beta(x)\beta(x^{-1})\beta(y) = \beta(y)\beta(x)\beta(x^{-1})$$

and so

$$\beta(x)\beta(x^{-1})\beta(y) = \beta(y)\beta(x)\beta(x^{-1})$$

for all $x, y$ in $G$.

Thus, for every $x \in G$, the element $\beta(x)\beta(x^{-1})$ commutes with every element of $\text{Im}(\beta)$, that is to say,

$$\text{Im}(\beta - \beta \alpha) \subseteq \text{Center of } \text{Im}(\beta),$$

as wanted.

3. In particular, let us set $\beta = \varepsilon$ (identity operator).

One has clearly $\text{Im}(\varepsilon) = G$ and, since the condition

$$\text{Im}(\varepsilon - \varepsilon \alpha) \subseteq \text{Center of } G$$

means that

$$x \alpha(x^{-1}) \in \text{Center of } G$$

for every $x \in G$, one obtains

**Corollary 1.** If $\alpha$ is an endomorphism of $G$, then the quotient group $G/\text{Ker}(\varepsilon - \alpha)$ is abelian, if and only if, for every $x \in G$, $x \alpha(x^{-1})$ is in the center of $G$.

This Corollary is the Theorem 3 in [1].

Now, let us set $\beta = \varepsilon$.

Then, if the endomorphism $\alpha$ is normal, i.e., if

$$(3) \quad \alpha(uvur^{-1}) = u\alpha(v)ur^{-1} \quad \text{for all } u, v \text{ in } G,$$

one sees that the condition (2) holds.

Indeed, from (3) it follows, by setting $u = \alpha(x^{-1})$ and $v = y$,

$$\alpha^2(x^{-1})\alpha(y)\alpha^2(x) = \alpha(x^{-1})\alpha(y)\alpha(x)$$

that is to say,

$$\alpha(x)\alpha^2(x^{-1})\alpha(y) = \alpha(y)\alpha(x)\alpha^2(x^{-1})$$

for all $x, y$ in $G$.

This means that

$$(x - \alpha^2)(x)\alpha(y) = \alpha(y)(x - \alpha^2)(x)$$

for all $x, y$ in $G$ and so the condition (2) holds.

By Theorem above, the group $G/\text{Ker}(\varepsilon - \alpha^2)$ is abelian and one obtains the following result, stated in [1] as Theorem 2:

**Corollary 2.** If $\alpha$ is a normal endomorphism of $G$, then $\alpha - \alpha^2$ is also an endomorphism and $G/\text{Ker}(\alpha - \alpha^2)$ is abelian.

**BIBLIOGRAPHY**
