Geodesic curvature of a curve of a vector field

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1. Introduction.

Let \( V \) be a vector field in a surface in a Euclidean space of three dimensions. Pan [3] has studied the normal curvature of the vector field \( V \) with generalization obtained by the author [2].

The object of the present note is to define the geodesic curvature of the curve of the vector field and obtain some properties. As a special case when the curves of the vector field \( V \) form an orthogonal net of co-ordinate curves, these geodesic curvatures have the known form.

2. Consider upon a surface \( S \)

\[ x^i = x^i(u^1, u^2) \quad (i = 1, 2, 3), \]

a curve \( C \) defined by

\[ u^x = u^x(s) \quad (x = 1, 2). \]

With each point of the surface we associate an arbitrary but fixed vector field \( V \). The components \( v^i \) and \( p^x \) of the vector field \( V \) in the \( x \)'s and \( u \)'s are connected by the relation

\[ v^i = x^i_{,x} p^x. \]

A curve on the surface \( S \) along which the vectors of the vector field \( V \) are tangent is called the curve of the vector field. It is defined by [3]

\[ \varepsilon_{x\beta} p^x \, du^\beta = 0. \]

The geodesic curvature of the curve \( C_V \) of the vector field \( V \) shall, therefore, be defined by [1]

\[ (2.1) v^k g = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\beta} \left( \frac{\sqrt{g} \, g^{\alpha\beta} \, \varepsilon_{x\lambda} p^\lambda}{(g^{\alpha\beta} \varepsilon_{x\lambda} \varepsilon_{x\mu} p^\lambda p^\mu)^{1/2}} \right). \]

Use of formulae

\[ (2.2) \quad g^{\alpha\beta} \varepsilon_{x\lambda} \varepsilon_{x\mu} = g_{\gamma\delta} \]

\[ (2.3) \quad \varepsilon_{x\lambda} g^{\beta\alpha} = \varepsilon_{x\lambda} g_{\gamma\delta} \]

in (2.1) yields the relation

\[ v^k g = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^\beta} \left( \frac{\varepsilon_{x\lambda} p^\lambda}{(g_{x\mu} p^\alpha p^\mu)^{1/2}} \right). \]

In particular when \( p^x \) are the components of a unit vector, we have

\[ (2.4) \quad -v^k g = \varepsilon_{x\lambda} p_{x,\beta} \]

where semi-colon (;) followed by an index denotes covariant differentiation with respect to \( u \) with that index. Since the right hand expression of (2.4) is a scalar called the curl of the vector \( p_\alpha \) [3], we have
“The geodesic curvature of the curve $C_V$ of a unit vector field $V$ is a scalar which numerically equals the curl of the covariant components of the vector field $V$.”

Evidently this curl vanishes if $p_\alpha$ are the components of the gradient of a scalar $\phi$. Therefore

“The necessary and sufficient condition for the vanishing of the geodesic curvature of a curve of a vector field is that the vector be a gradient”.

Suppose now that the vectors of the unit vector field $V$ form an orthogonal net of co-ordinate curves, then from (2.4) we obtain for the geodesic curvatures $v^k g_1$, and $v^k g_2$ of these curves the following relations

$$(2.5) \quad v^k g_1 = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^1} (g_{11} p^1)$$

$$(2.6) \quad v^k g_2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^1} (g_{22} p^2).$$

It is well to recall that

$$p^\alpha = \frac{d u^\alpha}{d s} \quad \alpha = 1, 2$$

and

$$\frac{d v^1}{d s} = \frac{1}{\sqrt{g_{11}}} \frac{d u^2}{d s} = -\frac{1}{\sqrt{g_{22}}}$$

to verify that (2.5) and (2.6) are the known results for the geodesic curvature of these curves.

Next we consider the orthogonal trajectory $C_W$ of the curve $C_V$ of the vector field $V$. These are defined by

$$\varepsilon_{\rho \lambda} g^{\rho \mu} \varepsilon_{\nu \alpha} p^\alpha d u^\nu = 0$$

which by virtue of (2.2) reduces to

$$g_{\alpha \lambda} p^\alpha d u^\lambda = 0.$$ 

If $w^k g$ denotes the geodesic curvature of the curve $C_W$, we have

$$w^k g = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^1} \left( \sqrt{g} g^{\alpha \beta} g_{\lambda \alpha} p^\alpha \right)$$

which yields on simplification

$$(2.7) \quad w^k g = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^1} \left( \sqrt{g} p^{\beta} \right)$$

Assuming $p^\alpha$ to be a unit vector, we obtain from (2.7)

$$-w^k g = \text{div} p^\beta$$

Thus

“The geodesic curvature of the orthogonal trajectory of the curve $C_V$ of a unit vector field $V$ numerically equals the divergence of the vector field”.

We can arrive at the above result also by considering a unit vector field $w^i (= x_i \alpha q^\alpha)$ orthogonal to the unit vector field $v^i (= x_i \alpha p^\alpha)$.

From (2.1), the geodesic curvature of a curve $C_W$ of the vector field $W$ is given by

$$w^k g = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^1} \left( \varepsilon_{\gamma \beta} g_{\gamma \lambda} q^\lambda \right)$$

which on using the relations

$$\varepsilon_{\alpha \lambda} p^\alpha = q_\lambda$$

$$\varepsilon_{\beta} \varepsilon_{\gamma \mu} = \delta^\beta_\mu$$

yields

$$w^k g = -\text{div} p^\beta.$$ 

Therefore

“The geodesic curvature of a curve of a unit vector field orthogonal to another unit vector field equals numerically the divergence of the latter”.

REFERENCES

